

The averaging of multi-dimensional Poisson brackets for systems having pseudo-phases

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*Dedicated to the 75th birthday
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Abstract

We consider features of the Hamiltonian formulation of the Whitham method in the presence of pseudo-phases. As we show, an analog of the procedure of averaging of the Poisson bracket with the reduced number of the first integrals can be suggested in this case. The averaged bracket gives a Poisson structure for the corresponding Whitham system having the form similar to the structures arising in the presence of ordinary phases.

1 Introduction. Hamiltonian structures in the Whitham method.

In this paper we consider the Hamiltonian formulation of the Whitham method for multi-dimensional systems having some additional property. Namely, we consider multi-dimensional systems which possess the so-called “pseudo-phases” having special physical or geometrical meaning. This property manifests itself in particular in the definition of the multi-phase solutions of the corresponding systems and in the form of the corresponding Whitham equations. Our considerations here will be devoted to the Hamiltonian formulation of the Whitham equations which will be connected with the procedure of the averaging of multi-dimensional Poisson structures in the presence of pseudo-phases. So, we consider the evolutionary systems

$$\varphi_t^i = F^i(\varphi, \varphi_x, \varphi_{xx}, \dots) \equiv F^i(\varphi, \varphi_{x^1}, \dots, \varphi_{x^d}, \dots) \quad (1.1)$$

$i = 1, \dots, n$, $\varphi = (\varphi^1, \dots, \varphi^n)$, with d spatial dimensions, and their m -phase solutions which are usually written in the form

$$\varphi^i(\mathbf{x}, t) = \Phi^i(\mathbf{k}_1(\mathbf{U})x^1 + \dots + \mathbf{k}_d(\mathbf{U})x^d + \omega(\mathbf{U})t + \boldsymbol{\theta}_0, \mathbf{U}) \quad (1.2)$$

with some 2π -periodic in each θ^α functions

$$\Phi^i(\boldsymbol{\theta}, \mathbf{U}) \equiv \Phi^i(\theta^1, \dots, \theta^m, \mathbf{U})$$

Here the functions $\mathbf{k}_q(\mathbf{U}) = (k_q^1(\mathbf{U}), \dots, k_q^m(\mathbf{U}))$ and $\boldsymbol{\omega}(\mathbf{U}) = (\omega^1(\mathbf{U}), \dots, \omega^m(\mathbf{U}))$ represent the “wave numbers” and the “frequencies” of the m -phase solutions. The parameters $\boldsymbol{\theta}_0$ represent the “initial phase shifts”, which can take arbitrary values on the family of the m -phase solutions.

Let us say also here that the function $f(\mathbf{x})$ represents a quasiperiodic function on \mathbb{R}^d with the wave numbers $(\mathbf{k}_1, \dots, \mathbf{k}_d)$ if it comes from a smooth function $f(\boldsymbol{\theta})$ on the torus \mathbb{T}^m :

$$f(\mathbf{k}_1 x^1 + \dots + \mathbf{k}_d x^d + \boldsymbol{\theta}_0) \rightarrow f(x^1, \dots, x^d)$$

under the corresponding mapping $\mathbb{R}^d \rightarrow \mathbb{T}^m$.

Let us call a smooth family of m -phase solutions of (1.1) any family (1.2) with a smooth dependence of the functions $\Phi(\boldsymbol{\theta}, \mathbf{U})$ on some finite number of parameters $\mathbf{U} = (U^1, \dots, U^N)$.

Here, however, we will need to generalize the definition of m -phase solutions of system (1.1) to include the presence of the pseudo-phases in the consideration. Let us say that the method of pseudo-phases was introduced by Whitham in [47] in connection with the Lagrangian structure of the Whitham system for the Korteweg - de Vries (KdV) equation. In this paper the appearance of pseudo-phases will be connected with the geometrical or physical meaning of the field variables $(\varphi^1, \dots, \varphi^n)$. Namely, it appears quite often that a part of the variables $(\varphi^1, \dots, \varphi^n)$ represents in fact some geometrical (or physical) “phase” variables growing linearly with the spatial or time variables. Thus, we have to separate the variables $(\varphi^1, \dots, \varphi^n)$ into two parts

$$(\varphi^1, \dots, \varphi^n) = (\rho^1, \dots, \rho^{n_1}, \phi^1, \dots, \phi^{n_2}) \quad , \quad (n_1 + n_2 = n) \quad , \quad (1.3)$$

representing the “density-type” and the “phase-type” variables respectively. Now, we will put slightly different conditions for the variables $(\rho^1, \dots, \rho^{n_1})$ and $(\phi^1, \dots, \phi^{n_2})$ in the definition of the m -phase solutions of (1.1) putting

$$\rho^i(\mathbf{x}, t) = R^i(\mathbf{k}_1(\mathbf{U}) x^1 + \dots + \mathbf{k}_d(\mathbf{U}) x^d + \boldsymbol{\omega}(\mathbf{U}) t + \boldsymbol{\theta}_0, \mathbf{U}) \quad , \quad i = 1, \dots, n_1 \quad (1.4)$$

$$\begin{aligned} \phi^j(\mathbf{x}, t) = & \Psi^j(\mathbf{k}_1(\mathbf{U}) x^1 + \dots + \mathbf{k}_d(\mathbf{U}) x^d + \boldsymbol{\omega}(\mathbf{U}) t + \boldsymbol{\theta}_0, \mathbf{U}) + \\ & + p_1^j(\mathbf{U}) x^1 + \dots + p_d^j(\mathbf{U}) x^d + \Omega^j(\mathbf{U}) t + \tau_0^j \quad , \quad j = 1, \dots, n_2 \end{aligned} \quad (1.5)$$

with some 2π -periodic in each θ^α functions $\mathbf{R}(\boldsymbol{\theta}, \mathbf{U})$, $\Psi(\boldsymbol{\theta}, \mathbf{U})$.

It is natural to put also the normalization

$$\int_0^{2\pi} \dots \int_0^{2\pi} \Psi^j(\boldsymbol{\theta} + \boldsymbol{\theta}_0, \mathbf{U}) \frac{d^m \boldsymbol{\theta}}{(2\pi)^m} \equiv 0 \quad , \quad j = 1, \dots, n_2 \quad (1.6)$$

According to the meaning of the variables $\phi^j(\mathbf{x}, t)$, only their spatial or time derivatives have in fact the physical sense, so, the right-hand part of system (1.1) in the variables $(\boldsymbol{\rho}, \boldsymbol{\phi})$ should contain only the spatial derivatives of $\phi^j(\mathbf{x}, t)$. We can rewrite then the initial system (1.1) in the variables $(\boldsymbol{\rho}, \boldsymbol{\phi})$ in the form

$$\begin{aligned} \rho_t^i &= A^i(\boldsymbol{\rho}, \boldsymbol{\rho}_x, \boldsymbol{\phi}_x, \boldsymbol{\rho}_{xx}, \boldsymbol{\phi}_{xx}, \dots) \quad , \quad i = 1, \dots, n_1 \\ \phi_t^j &= B^j(\boldsymbol{\rho}, \boldsymbol{\rho}_x, \boldsymbol{\phi}_x, \boldsymbol{\rho}_{xx}, \boldsymbol{\phi}_{xx}, \dots) \quad , \quad j = 1, \dots, n_2 \end{aligned} \quad (1.7)$$

The functions $R^i(\boldsymbol{\theta}, \mathbf{U})$ and $\Psi^j(\boldsymbol{\theta}, \mathbf{U})$ are then defined by the system

$$\begin{aligned} \omega^\alpha R_{\theta^\alpha}^i - A^i \left(\mathbf{R}, k_1^{\beta_1} \mathbf{R}_{\theta^{\beta_1}}, \dots, k_d^{\beta_d} \mathbf{R}_{\theta^{\beta_d}}, k_1^{\gamma_1} \boldsymbol{\Psi}_{\theta^{\gamma_1}} + \mathbf{p}_1, \dots \right) &= 0 \\ \Omega^j + \omega^\alpha \Psi_{\theta^\alpha}^j - B^j \left(\mathbf{R}, k_1^{\beta_1} \mathbf{R}_{\theta^{\beta_1}}, \dots, k_d^{\beta_d} \mathbf{R}_{\theta^{\beta_d}}, k_1^{\gamma_1} \boldsymbol{\Psi}_{\theta^{\gamma_1}} + \mathbf{p}_1, \dots \right) &= 0 \end{aligned} \quad (1.8)$$

(summation over repeated indexes) with normalization conditions (1.6).

In this paper we will need in fact to put more special requirements to the definition of the pseudo-phases in the general Whitham scheme. In particular, we will assume in this paper that the values $(\mathbf{k}_1, \dots, \mathbf{k}_d, \boldsymbol{\omega}, \mathbf{p}_1, \dots, \mathbf{p}_d, \boldsymbol{\Omega})$ represent independent parameters on the family Λ of m -phase solutions of (1.7) such that the number of the parameters \mathbf{U} on Λ is not less than $m(1+d) + n_2(1+d)$. Thus, we will assume here that the family Λ has $N = m(1+d) + n_2(1+d) + s$, ($s \geq 0$) parameters except the initial phase shifts, which can be chosen in the form

$$(U^1, \dots, U^N) = (\mathbf{k}_1, \dots, \mathbf{k}_d, \boldsymbol{\omega}, \mathbf{p}_1, \dots, \mathbf{p}_d, \boldsymbol{\Omega}, n^1, \dots, n^s)$$

where (n^1, \dots, n^s) are some additional parameters in the set (U^1, \dots, U^N) (if any). In general, the parameters (U^1, \dots, U^N) can be chosen in different ways, we just assume that they do not change under the initial phase shifts on Λ . The parameters τ_0^j , $j = 1, \dots, n_2$, as well as θ_0^α , $\alpha = 1, \dots, m$ represent the initial phase shifts on the family Λ .

Another important requirement on the pseudo-phases in our scheme will be considered in the next chapter and is connected with the Hamiltonian structure of system (1.1).

As it is well known, in the Whitham approach ([45, 46, 47]) the parameters (U^1, \dots, U^N) become “slow” functions of coordinates and time. More precisely, we have to make the coordinate change $x^q \rightarrow X^q = \epsilon x^q$, $t \rightarrow T = \epsilon t$, $\epsilon \rightarrow 0$ and introduce the slow functions $S^\alpha(\mathbf{X}, T)$, $\alpha = 1, \dots, m$, $\Sigma^j(\mathbf{X}, T)$, $j = 1, \dots, n_2$. We try to construct then the asymptotic solutions of the system

$$\begin{aligned} \epsilon \rho_T^i &= A^i(\boldsymbol{\rho}, \epsilon \boldsymbol{\rho}_\mathbf{X}, \epsilon \boldsymbol{\phi}_\mathbf{X}, \epsilon^2 \boldsymbol{\rho}_{\mathbf{X}\mathbf{X}}, \epsilon^2 \boldsymbol{\phi}_{\mathbf{X}\mathbf{X}}, \dots) \quad , \quad i = 1, \dots, n_1 \\ \epsilon \phi_T^j &= B^j(\boldsymbol{\rho}, \epsilon \boldsymbol{\rho}_\mathbf{X}, \epsilon \boldsymbol{\phi}_\mathbf{X}, \epsilon^2 \boldsymbol{\rho}_{\mathbf{X}\mathbf{X}}, \epsilon^2 \boldsymbol{\phi}_{\mathbf{X}\mathbf{X}}, \dots) \quad , \quad j = 1, \dots, n_2 \end{aligned} \quad (1.9)$$

with the main term having the form

$$\begin{aligned} \rho_{(0)}^i &= R^i \left(\frac{\mathbf{S}(\mathbf{X}, T)}{\epsilon} + \boldsymbol{\theta}_0(\mathbf{X}, T) + \boldsymbol{\theta}, \mathbf{U}(\mathbf{X}, T) \right) \quad , \quad i = 1, \dots, n_1, \\ \phi_{(0)}^j &= \Psi^j \left(\frac{\mathbf{S}(\mathbf{X}, T)}{\epsilon} + \boldsymbol{\theta}_0(\mathbf{X}, T) + \boldsymbol{\theta}, \mathbf{U}(\mathbf{X}, T) \right) + \frac{1}{\epsilon} \Sigma^j(\mathbf{X}, T) + \tau_0^j(\mathbf{X}, T) \quad , \quad j = 1, \dots, n_2. \end{aligned} \quad (1.10)$$

Substituting the functions from Λ it is easy to get the relations

$$S_T^\alpha = \omega^\alpha(\mathbf{U}) \quad , \quad S_{X^q}^\alpha = k_q^\alpha(\mathbf{U}) \quad , \quad \Sigma_T^j = \Omega^j(\mathbf{U}) \quad , \quad \Sigma_{X^q}^j = p_q^j(\mathbf{U})$$

in the zero approximation, which gives the compatibility conditions

$$k_{qT}^\alpha = \omega_{X^q}^\alpha \quad , \quad p_{qT}^j = \Omega_{X^q}^j \quad , \quad k_{qX^p}^\alpha = k_{pX^q}^\alpha \quad , \quad p_{qX^k}^j = p_{kX^q}^j \quad (1.11)$$

for the parameters $(\mathbf{k}_1, \dots, \mathbf{k}_d, \boldsymbol{\omega}, \mathbf{p}_1, \dots, \mathbf{p}_d, \boldsymbol{\Omega})$ on the family Λ .

The second part of restrictions on the parameters (U^1, \dots, U^N) in the Whitham method is given by the requirement of the existence of the first correction $(\boldsymbol{\rho}_{(1)}, \boldsymbol{\phi}_{(1)})$ to solution (1.10)

$$\begin{aligned}\rho^i &\simeq \rho_{(0)}^i + \epsilon \rho_{(1)}^i \left(\frac{\mathbf{S}(\mathbf{X}, T)}{\epsilon} + \boldsymbol{\theta}, \mathbf{X}, T \right) \\ \phi^i &\simeq \phi_{(0)}^i + \epsilon \phi_{(1)}^i \left(\frac{\mathbf{S}(\mathbf{X}, T)}{\epsilon} + \boldsymbol{\theta}, \mathbf{X}, T \right)\end{aligned}$$

on the space of 2π -periodic in $\boldsymbol{\theta}$ functions (see [26]).

The functions $(\boldsymbol{\rho}_{(1)}(\boldsymbol{\theta}, \mathbf{X}, T), \boldsymbol{\phi}_{(1)}(\boldsymbol{\theta}, \mathbf{X}, T))$ are defined by the linear system

$$\hat{L}_{[\mathbf{U}(\mathbf{X}, T), \boldsymbol{\theta}_0(\mathbf{X}, T)]} \begin{pmatrix} \boldsymbol{\rho}_{(1)}(\boldsymbol{\theta}, \mathbf{X}, T) \\ \boldsymbol{\phi}_{(1)}(\boldsymbol{\theta}, \mathbf{X}, T) \end{pmatrix} = \mathbf{f}_1(\boldsymbol{\theta}, \mathbf{X}, T)$$

where $\hat{L}_{[\mathbf{U}(\mathbf{X}, T), \boldsymbol{\theta}_0(\mathbf{X}, T)]}$ is the linear operator given by the linearization of the left-hand part of system (1.8) on the corresponding functions from Λ and $\mathbf{f}_1(\boldsymbol{\theta}, \mathbf{X}, T)$ is the first ϵ -discrepancy defined after the substitution of (1.10) in (1.9).

The operator $\hat{L}_{[\mathbf{U}(\mathbf{X}, T), \boldsymbol{\theta}_0(\mathbf{X}, T)]}$ represents a differential in $\boldsymbol{\theta}$ operator with periodic coefficients at every fixed \mathbf{X} and T . We get then that the second part of the Whitham system should be given by the orthogonality of the function $\mathbf{f}_1(\boldsymbol{\theta}, \mathbf{X}, T)$ to all the left eigen-vectors of \hat{L} (the eigen-vectors of the adjoint operator) corresponding to the zero eigen-values at every fixed (\mathbf{X}, T) .

We should say, however, that the orthogonality of $\mathbf{f}_1(\boldsymbol{\theta}, \mathbf{X}, T)$ to all the left eigen-vectors of \hat{L} with zero eigen-values is imposed usually just in the one-phase situation. In this case we have usually just a finite number of such eigen-vectors depending regularly on the parameters (U^1, \dots, U^N) . The corresponding orthogonality conditions together with conditions (1.11) give then a regular system of hydrodynamic type which represents the Whitham system in the one-phase situation. Another important thing taking place in the one-phase situation is the possibility of constructing of all the corrections $\boldsymbol{\varphi}_{(n)}$ in all orders of ϵ and representing the asymptotic solution as a regular series in integer powers of ϵ .

This situation, however, does not usually take place in the multi-phase case where the behavior of the eigen-vectors of \hat{L} is usually much more complicated. Thus, the kernels of the operators \hat{L} and \hat{L}^\dagger depend usually in highly nontrivial way on the parameters \mathbf{U} , being finite- or infinite-dimensional for different values of (U^1, \dots, U^N) . In this situation it is natural to define the “regular” orthogonality conditions just by the requirement of orthogonality of \mathbf{f}_1 to the “regular” set of the kernel vectors of \hat{L}^\dagger which is usually finite also in the multi-phase case. Thus, we assume here that the kernels of the operators \hat{L} and \hat{L}^\dagger contain just a finite number of linearly independent “regular” eigen-vectors, i.e. the eigen-vectors smoothly depending on the parameters \mathbf{U} . The “regular” Whitham system is defined in this situation by conditions (1.11) and the orthogonality of the discrepancy $\mathbf{f}_1(\boldsymbol{\theta}, \mathbf{X}, T)$ to all the regular left eigen-vectors of \hat{L} corresponding to the zero eigen-value.

Let us say that the first correction $\boldsymbol{\varphi}_{(1)}$ to the asymptotic solution (1.10) can not be found here in such a simple form as in the one-phase situation. However, as the investigations of this situation show, the corrections to the main approximation $\boldsymbol{\varphi}_{(0)}$ still vanish as $\epsilon \rightarrow 0$ even in the multi-phase case (see [5, 6, 7]). So, despite the high non-triviality of the next approximation in this case ([5, 6, 7]), the regular Whitham system still plays very important role in consideration of slow-modulated m -phase solutions.

It is not difficult to see that the Whitham system imposes restrictions just on the functions $\mathbf{U}(\mathbf{X}, T)$ and does not contain the parameters $\boldsymbol{\theta}_0(\mathbf{X}, T)$ and $\boldsymbol{\tau}_0(\mathbf{X}, T)$. Indeed, the functions $\boldsymbol{\theta}_0(\mathbf{X}, T)$ and $\boldsymbol{\tau}_0(\mathbf{X}, T)$ can be considered just as ϵ -corrections to the functions $\mathbf{S}(\mathbf{X}, T)$ and $\boldsymbol{\Sigma}(\mathbf{X}, T)$, so the constraints arising on the first step include just the main terms $\mathbf{S}(\mathbf{X}, T)$ and $\boldsymbol{\Sigma}(\mathbf{X}, T)$, while the restrictions on $\boldsymbol{\theta}_0(\mathbf{X}, T)$ and $\boldsymbol{\tau}_0(\mathbf{X}, T)$ arise in the higher approximations (if they exist) (see [26]).¹

For the correct construction of the modulated solutions and a good definition of the Whitham system we have to require in fact one more thing from the family Λ . Namely, the correct procedure of constructing of modulated solutions can be implemented on the “complete regular families” Λ of m -phase solutions of (1.7). Let us give here the corresponding definition. Let us consider the set of parameters \mathbf{U} in the form

$$\mathbf{U} = (\mathbf{k}_1, \dots, \mathbf{k}_d, \boldsymbol{\omega}, \mathbf{p}_1, \dots, \mathbf{p}_d, \boldsymbol{\Omega}, n^1, \dots, n^s)$$

It is easy to see then that the vectors

$$\begin{aligned} \boldsymbol{\xi}_{(\alpha)[\mathbf{U}, \boldsymbol{\theta}_0]} &= (\mathbf{R}_{\theta^\alpha}(\boldsymbol{\theta} + \boldsymbol{\theta}_0, \mathbf{U}), \boldsymbol{\Psi}_{\theta^\alpha}(\boldsymbol{\theta} + \boldsymbol{\theta}_0, \mathbf{U}))^t, \quad \alpha = 1, \dots, m, \\ \boldsymbol{\eta}_{(l)[\mathbf{U}, \boldsymbol{\theta}_0]} &= (\mathbf{R}_{n^l}(\boldsymbol{\theta} + \boldsymbol{\theta}_0, \mathbf{U}), \boldsymbol{\Psi}_{n^l}(\boldsymbol{\theta} + \boldsymbol{\theta}_0, \mathbf{U}))^t, \quad l = 1, \dots, s, \\ \text{and } \boldsymbol{\zeta}_{(j)[\mathbf{U}, \boldsymbol{\theta}_0]} &= (0, \dots, 1, \dots, 0)^t \quad ((n_1 + j)\text{th position}), \quad j = 1, \dots, n_2 \end{aligned}$$

represent regular (right) eigen-vectors of the operators $\hat{L}_{[\mathbf{U}, \boldsymbol{\theta}_0]}$ corresponding to the zero eigen-value.

Definition 1.1.

We call family Λ a complete regular family of m -phase solutions of (1.7) with n_2 pseudo-phases if:

1) The values $\mathbf{k}_p = (k_p^1, \dots, k_p^m)$, $\boldsymbol{\omega} = (\omega^1, \dots, \omega^m)$, $\mathbf{p}_q = (p_q^1, \dots, p_q^{n_2})$, and $\boldsymbol{\Omega} = (\Omega^1, \dots, \Omega^{n_2})$ are all independent, such that the total set of independent parameters on Λ can be represented in the form

$$(\mathbf{U}, \boldsymbol{\theta}_0, \boldsymbol{\tau}_0) = (\mathbf{k}_1, \dots, \mathbf{k}_d, \boldsymbol{\omega}, \mathbf{p}_1, \dots, \mathbf{p}_d, \boldsymbol{\Omega}, n^1, \dots, n^s, \boldsymbol{\theta}_0, \boldsymbol{\tau}_0)$$

2) The vectors $\boldsymbol{\xi}_{(\alpha)[\mathbf{U}, \boldsymbol{\theta}_0]}$, $\boldsymbol{\eta}_{(l)[\mathbf{U}, \boldsymbol{\theta}_0]}$, and $\boldsymbol{\zeta}_{(j)[\mathbf{U}, \boldsymbol{\theta}_0]}$ are linearly independent and represent the maximal linearly independent set of the kernel vectors of $\hat{L}_{[\mathbf{U}, \boldsymbol{\theta}_0]}$ smoothly depending on the parameters \mathbf{U} ;

3) The operator $\hat{L}_{[\mathbf{U}, \boldsymbol{\theta}_0]}$ also has exactly $m + s + n_2$ linearly independent left eigen-vectors corresponding to the zero eigen-value

$$\boldsymbol{\kappa}_{[\mathbf{U}]}^{(q)}(\boldsymbol{\theta} + \boldsymbol{\theta}_0) = \boldsymbol{\kappa}_{[\mathbf{k}_1, \dots, \mathbf{k}_d, \boldsymbol{\omega}, \mathbf{p}_1, \dots, \mathbf{p}_d, \boldsymbol{\Omega}, \mathbf{n}]}^{(q)}(\boldsymbol{\theta} + \boldsymbol{\theta}_0), \quad q = 1, \dots, m + s + n_2,$$

defined for all values of \mathbf{U} and depending smoothly on the parameters \mathbf{U} .

Let us call the regular Whitham system for a complete regular family of m -phase solutions of system (1.7) with n_2 pseudo-phases the conditions of orthogonality of the discrepancy $\mathbf{f}_{(1)}(\boldsymbol{\theta}, \mathbf{X}, T)$ to the vectors $\boldsymbol{\kappa}_{[\mathbf{U}(\mathbf{X}, T)]}^{(q)}(\boldsymbol{\theta} + \boldsymbol{\theta}_0(\mathbf{X}, T))$

$$\int_0^{2\pi} \dots \int_0^{2\pi} \kappa_{[\mathbf{U}(\mathbf{X}, T)]}^{(q)}(\boldsymbol{\theta} + \boldsymbol{\theta}_0(\mathbf{X}, T)) f_{(1)}^i(\boldsymbol{\theta}, \mathbf{X}, T) \frac{d^m \theta}{(2\pi)^m} = 0 \quad (1.12)$$

¹A more detailed discussion of the phase shift $\boldsymbol{\theta}_0(\mathbf{X}, T)$ can be found for example in [21, 22, 31, 8]. We should note also that the phase shift can play rather important role in the weakly nonlinear case, leading to nontrivial corrections to the Whitham system ([38], see also [32, 8]).

($q = 1, \dots, m + s + n_2$) and the compatibility conditions

$$k_{pT}^\alpha = \omega_{X^p}^\alpha \quad , \quad p_{lT}^j = \Omega_{X^l}^j \quad (1.13)$$

$$k_{pX^l}^\alpha = k_{lX^p}^\alpha \quad , \quad p_{lX^k}^j = p_{kX^l}^j \quad (1.14)$$

$\alpha = 1, \dots, m, \quad p, l, k = 1, \dots, d, \quad j = 1, \dots, n_2$.

For our further purposes it will be convenient to separate the evolutionary part of the Whitham system and purely spatial constraints. So, let us call here relations (1.12) - (1.13) the evolutionary part of a regular Whitham system. The relations (1.14) will be considered then as additional constraints for the evolutionary system (1.12) - (1.13).

The evolutionary part of a regular Whitham system provides exactly $m(d+1) + n_2(d+1) + s$ independent relations for $N = m(d+1) + n_2(d+1) + s$ parameters $\mathbf{U} = (\mathbf{k}_1, \dots, \mathbf{k}_d, \boldsymbol{\omega}, \mathbf{p}_1, \dots, \mathbf{p}_d, \boldsymbol{\Omega}, \mathbf{n})$ at every \mathbf{X} and T . We can assume also, that in generic case the derivatives \mathbf{U}_T can be expressed in terms of the spatial derivatives \mathbf{U}_{X^l} , such that we can write the evolutionary part of a regular Whitham system in the form

$$U_T^\nu = V_\mu^{\nu l}(\mathbf{U}) U_{X^l}^\mu \quad (1.15)$$

for general set of parameters \mathbf{U} .

Following B.A. Dubrovin and S.P. Novikov we will call systems having the form (1.15) the systems of Hydrodynamic Type in d spatial dimensions.

The Hamiltonian theory of systems (1.15) was started by B.A. Dubrovin and S.P. Novikov who introduced the concept of the Poisson bracket of Hydrodynamic Type ([9, 10, 11, 12]). The local Poisson brackets of Hydrodynamic Type (Dubrovin - Novikov brackets) can be represented by the following general form

$$\{U^\nu(\mathbf{X}), U^\mu(\mathbf{Y})\} = g^{\nu\mu l}(\mathbf{U}(\mathbf{X})) \delta_{X^l}(\mathbf{X} - \mathbf{Y}) + b_\lambda^{\nu\mu l}(\mathbf{U}(\mathbf{X})) U_{X^l}^\lambda \delta(\mathbf{X} - \mathbf{Y}) \quad (1.16)$$

(summation over repeated indexes).

The theory of brackets (1.16) is best developed in the case of one spatial ($d = 1$) dimension. Thus, expression (1.16) with non-degenerate tensor $g^{\nu\mu}$ defines a Poisson bracket for $d = 1$ if and only if the tensor $g^{\nu\mu}(\mathbf{U})$ represents a flat pseudo-Riemannian (contravariant) metric on the space of parameters \mathbf{U} , while the functions $\Gamma_{\mu\gamma}^\nu(\mathbf{U}) = -g_{\mu\lambda}(\mathbf{U}) b_\gamma^{\lambda\nu}(\mathbf{U})$ ($g^{\nu\lambda}(\mathbf{U}) g_{\lambda\mu}(\mathbf{U}) \equiv \delta_\mu^\nu$) represent the corresponding Christoffel symbols. As a corollary, every Dubrovin - Novikov bracket in one-dimensional case can be written in the canonical (constant) form

$$\{c^\nu(X), c^\mu(Y)\} = e^\nu \delta^{\nu\mu} \delta'(X - Y) \quad , \quad e^\nu = \pm 1$$

using the flat coordinates $c^\nu = c^\nu(\mathbf{U})$ of the metric $g_{\nu\mu}(\mathbf{U})$.

It's not difficult to see also that the functionals

$$C^\nu = \int_{-\infty}^{+\infty} c^\nu(X) dX \quad , \quad P = \int_{-\infty}^{+\infty} \frac{1}{2} \sum_{\nu=1}^N e^\nu (c^\nu)^2(X) dX$$

represent annihilators and the momentum functional of bracket (1.16) for the case $d = 1$. The systems of Hydrodynamic Type are generated by the functionals of Hydrodynamic Type

$$H = \int_{-\infty}^{+\infty} h(\mathbf{U}) dX$$

according to the Dubrovin - Novikov bracket.

The Hamiltonian approach plays very important role in the theory of integrability of the Hydrodynamic Type systems in the case of one spatial dimension. Thus, according to the conjecture of S.P. Novikov, any system of Hydrodynamic Type which can be written in the diagonal form

$$U_T^\nu = V^\nu(\mathbf{U}) U_X^\nu$$

and is Hamiltonian with respect to some local bracket of Hydrodynamic Type is integrable. The Novikov conjecture was proved by S.P. Tsarev ([43, 44]), who also suggested a method of integration of systems of this kind. The method suggested by Tsarev (the generalized hodograph method) can be applied in fact to a wider class of “semi-Hamiltonian” systems, which contains all the diagonalizable Hamiltonian systems as a subclass. As was shown later, the class of “semi-Hamiltonian systems” contains also the systems, Hamiltonian with respect to the Mokhov - Ferapontov bracket ([35]) or the Ferapontov brackets ([16, 17]), which can be considered as the weakly nonlocal generalizations of the Dubrovin - Novikov bracket. Let us give here the references on papers [35, 16, 17, 18, 19, 41, 30] where the detailed discussion of the weakly nonlocal Poisson structures can be found.

Let us say, that the theory of the Dubrovin - Novikov brackets in the multi-dimensional case is more complicated than in the case $d = 1$. The most general properties of the multi-dimensional brackets (1.16) were investigated in [10, 36, 37]. However, the investigation of the brackets (1.16) in $d > 1$ dimensions still represents one of the most interesting branch of the theory of infinite-dimensional Poisson structures.

The Hamiltonian formulation of the Whitham method was also suggested by B.A. Dubrovin and S.P. Novikov who introduced the procedure of “averaging” of Hamiltonian structures in the theory of slow modulations ([9, 11, 12]). This approach is connected with the Whitham method for the evolutionary systems

$$\varphi_t^i = F^i(\varphi, \varphi_x, \dots)$$

having a local field-theoretic Poisson structure

$$\{\varphi^i(x), \varphi^j(y)\} = \sum_{k \geq 0} B_{(k)}^{ij}(\varphi, \varphi_x, \dots) \delta^{(k)}(x - y)$$

with the local Hamiltonian of the form

$$H = \int P_H(\varphi, \varphi_x, \dots) dx$$

The procedure of averaging of local field-theoretic Poisson brackets was first developed in the case of one spatial dimension and gives a local Poisson structure of Hydrodynamic Type for the corresponding Whitham system. The method of B.A. Dubrovin and S.P. Novikov is connected with the conservative form of the Whitham system and is based on the existence of a set of commuting local integrals

$$I^\nu = \int P^\nu(\varphi, \varphi_x, \dots) dx$$

which number is equal to the number of parameters U^ν on the family Λ .

The integrals I^ν should satisfy the relations

$$\{I^\nu, H\} = 0 \quad , \quad \{I^\nu, I^\mu\} = 0 \quad ,$$

such that we can write for the time evolution of the densities $P^\nu(\varphi, \varphi_x, \dots)$:

$$P_t^\nu(\varphi, \varphi_x, \dots) \equiv Q_x^\nu(\varphi, \varphi_x, \dots)$$

with some functions $Q^\nu(\varphi, \varphi_x, \dots)$. In the same way, the calculation of the Poisson brackets of the densities P^ν gives the relations

$$\{P^\nu(x), P^\mu(y)\} = \sum_{k \geq 0} A_k^{\nu\mu}(\varphi, \varphi_x, \dots) \delta^{(k)}(x - y)$$

where

$$A_0^{\nu\mu}(\varphi, \varphi_x, \dots) \equiv \partial_x Q^{\nu\mu}(\varphi, \varphi_x, \dots)$$

with some local functions $Q^{\nu\mu}(\varphi, \varphi_x, \dots)$.

It is natural to define the procedure $\langle \dots \rangle$ of averaging of any expression $f(\varphi, \varphi_x, \dots)$ over the phase variables on Λ putting

$$\langle f \rangle = \int_0^{2\pi} \dots \int_0^{2\pi} f(\Phi, k^\alpha \Phi_{\theta^\alpha}, \dots) \frac{d^m \theta}{(2\pi)^m}$$

The Dubrovin - Novikov bracket on the space of functions $\mathbf{U}(\mathbf{X})$, where $U^\nu \equiv \langle P^\nu \rangle$, is defined by the formula

$$\{U^\nu(X), U^\mu(Y)\} = \langle A_1^{\nu\mu} \rangle(\mathbf{U}) \delta'(X - Y) + \frac{\partial \langle Q^{\nu\mu} \rangle}{\partial U^\gamma} U_X^\gamma \delta(X - Y) \quad (1.17)$$

The Whitham system can be written now in the form

$$\langle P^\nu \rangle_T = \langle Q^\nu \rangle_X \quad , \quad \nu = 1, \dots, N$$

and can be proved to be Hamiltonian with respect to the bracket (1.17) with the Hamiltonian of Hydrodynamic Type

$$H_{av} = \int_{-\infty}^{+\infty} \langle P_H \rangle(\mathbf{U}(X)) dX$$

The Jacobi identity for bracket (1.17) was first proved in [28] using some regularity assumptions about the family Λ . A more detailed consideration of the justification of the Dubrovin - Novikov procedure in the single-phase and the multi-phase situations was presented in [33]. In particular, it was first shown in [33] that the justification of the procedure can be done also in the presence of “resonances” which can arise in the multi-phase situation. Let us note, that in [27] it was shown also that the method of averaging of the Lagrangian functional ([47]) can be also considered in terms of the Dubrovin - Novikov procedure for a wide class of local Lagrangian systems. In [29] the generalization of the Dubrovin - Novikov procedure for the weakly nonlocal brackets was also suggested.

Let us say that the investigation of the Hamiltonian properties of the Whitham systems was of a great interest since the pioneer works of B.A. Dubrovin and S.P. Novikov. Besides that, the general theory of the Dubrovin - Novikov brackets appeared to be extremely important in many subjects. As the most striking example, we can point out here the theory of the Frobenius manifolds built by B.A. Dubrovin and based on the theory of compatible Dubrovin - Novikov brackets (see e.g. [13, 14, 15]).

Among the papers devoted to the Hamiltonian structures of the Whitham systems we would like to cite here also the papers [42, 2] where the local and the weakly nonlocal Hamiltonian structures for the famous integrable hierarchies were considered.

Unfortunately, we can not present here the complete list of papers devoted to the Whitham approach. Let us just give here some incomplete list of classical papers where the fundamental aspects of the Whitham method were discussed [1, 3, 4, 5, 6, 7, 9, 11, 12, 20, 23, 24, 25, 26, 38, 39, 40, 45, 46, 47]. Let us say also, that we will discuss here just the Hamiltonian properties of the multi-dimensional Whitham systems in the case of the presence of the pseudo-phases.

In paper [34] the procedure of averaging of multi-dimensional local field-theoretic Poisson brackets was suggested. The approach used in [34] can be considered as a generalization of the Dubrovin - Novikov procedure to the multi-dimensional case. According to the approach of [34] we consider the regular Whitham system for a complete regular family Λ of m -phase solutions of system (1.1), parametrized by the values $(\mathbf{k}_1, \dots, \mathbf{k}_d, \boldsymbol{\omega}, \mathbf{n}, \boldsymbol{\theta}_0)$. We assume now that system (1.1) is Hamiltonian with respect to a local field-theoretic Poisson bracket

$$\{\varphi^i(\mathbf{x}), \varphi^j(\mathbf{y})\} = \sum_{l_1, \dots, l_d} B_{(l_1, \dots, l_d)}^{ij}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_{\mathbf{x}}, \dots) \delta^{(l_1)}(x^1 - y^1) \dots \delta^{(l_d)}(x^d - y^d) \quad (1.18)$$

$(l_1, \dots, l_d \geq 0)$, with a local Hamiltonian of the form

$$H = \int P_H(\boldsymbol{\varphi}, \boldsymbol{\varphi}_{\mathbf{x}}, \boldsymbol{\varphi}_{\mathbf{xx}}, \dots) d^d x \quad (1.19)$$

Like in the Dubrovin - Novikov procedure we have to require here the existence of N (equal to the number of parameters $(\mathbf{k}_1, \dots, \mathbf{k}_d, \boldsymbol{\omega}, \mathbf{n})$) first integrals

$$I^\nu = \int P^\nu(\boldsymbol{\varphi}, \boldsymbol{\varphi}_{\mathbf{x}}, \boldsymbol{\varphi}_{\mathbf{xx}}, \dots) d^d x \quad (1.20)$$

such that their values can be chosen as the parameters (U^1, \dots, U^N) on the family Λ . We assume also that all the integrals I^ν commute with each other and with the Hamiltonian H

$$\{I^\nu, I^\mu\} = 0 \quad , \quad \{I^\nu, H\} = 0 \quad (1.21)$$

according to bracket (1.18). For the time evolution of the densities $P^\nu(\mathbf{x})$ we can write

$$P_t^\nu(\boldsymbol{\varphi}, \boldsymbol{\varphi}_{\mathbf{x}}, \boldsymbol{\varphi}_{\mathbf{xx}}, \dots) = Q_{x^1}^{\nu 1}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_{\mathbf{x}}, \boldsymbol{\varphi}_{\mathbf{xx}}, \dots) + \dots + Q_{x^d}^{\nu d}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_{\mathbf{x}}, \boldsymbol{\varphi}_{\mathbf{xx}}, \dots)$$

with some functions $Q^{\nu l}$.

In fact, we have to put also some additional requirements on the family Λ and the set of the integrals I^ν . Namely, we have to require that the family Λ represents a regular Hamiltonian family of m -phase solutions of system (1.1) and the set (I^1, \dots, I^N) represents a complete Hamiltonian set of commuting integrals. So, we put in fact the following requirements:

1) The family Λ represents a complete regular family of m -phase solutions of system (1.1) according to Definition 1.1;

2) The bracket (1.18) has the same number of annihilators (N^1, \dots, N^s) on the space of the quasiperiodic functions with the wave numbers $(\mathbf{k}_1, \dots, \mathbf{k}_d)$ for every fixed values of $(\mathbf{k}_1, \dots, \mathbf{k}_d)$;

3) The values of the functionals (I^1, \dots, I^N) on the family Λ represent the set of parameters (U^1, \dots, U^N) on this family;

4) The Hamiltonian flows, generated by the functionals (I^1, \dots, I^N) according to bracket (1.18), generate on Λ linear phase shifts of θ_0 with frequencies $\omega^\nu(\mathbf{U})$, such that

$$\text{rk } ||\omega^{\alpha\nu}(\mathbf{U})|| = m$$

5) At every “point” of the “submanifold” Λ , having “coordinates” $(\mathbf{k}_1, \dots, \mathbf{k}_d, \boldsymbol{\omega}, \mathbf{n}, \boldsymbol{\theta}_0)$, the linear space generated by the variation derivatives $\delta I^\nu / \delta \varphi^i(\mathbf{x})$ contains the variation derivatives of all the corresponding annihilators N^q of the bracket (1.18), such that we can write

$$\left. \frac{\delta N^l}{\delta \varphi^i(\mathbf{x})} \right|_\Lambda = \sum_{\nu=1}^N \gamma_\nu^l(\mathbf{U}) \left. \frac{\delta I^\nu}{\delta \varphi^i(\mathbf{x})} \right|_\Lambda$$

for some functions $\gamma_\nu^l(\mathbf{U})$ on Λ .

Under the requirements formulated above the set (I^1, \dots, I^N) can be used for construction of a local field-theoretic Poisson bracket for the regular Whitham system on a regular Hamiltonian family Λ of m -phase solutions of system (1.1). The corresponding procedure in the absence of the pseudo-phases can be formulated in the following way:

The pairwise Poisson brackets of the densities $P^\nu(\mathbf{x})$, $P^\mu(\mathbf{y})$ can be represented in the form

$$\{P^\nu(\mathbf{x}), P^\mu(\mathbf{y})\} = \sum_{l_1, \dots, l_d} A_{l_1 \dots l_d}^{\nu\mu}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_\mathbf{x}, \dots) \delta^{(l_1)}(x^1 - y^1) \dots \delta^{(l_d)}(x^d - y^d)$$

($l_1, \dots, l_d \geq 0$). In the same way as in the one-dimensional case, we can also write here the relations

$$A_{0 \dots 0}^{\nu\mu}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_\mathbf{x}, \dots) \equiv \partial_{x^1} Q^{\nu\mu 1}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_\mathbf{x}, \dots) + \dots + \partial_{x^d} Q^{\nu\mu d}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_\mathbf{x}, \dots)$$

according to relations (1.21). Let us say, however, that the averaged Poisson bracket does not have in general the form (1.16) for $d > 1$, which is connected with the fact that the Hamiltonian structure should be defined now just on the “submanifold” in the space of functions $\mathbf{U}(\mathbf{X})$, given by the constraints $k_{qX^p}^\alpha = k_{pX^q}^\alpha$, $\alpha = 1, \dots, m$, $q, p = 1, \dots, d$. To define the corresponding Poisson bracket we have to introduce the coordinates $S^\alpha(\mathbf{X})$ ($\alpha = 1, \dots, m$) on this submanifold, defined by the relations $S_{X^q}^\alpha = k_q^\alpha(\mathbf{X})$. It is easy to see, that the spatial derivatives of the functions $S^\alpha(\mathbf{X})$ provide just md coordinates on the family Λ , connected with the wave numbers of the solutions. For the remaining $m + s$ coordinates we can use just arbitrary independent values U^γ , $\gamma = 1, \dots, m + s$ from the full set $U^\nu = \langle P^\nu \rangle$, $\nu = 1, \dots, N$ on Λ . The corresponding regular Whitham system on Λ can then be written in the form:

$$S_T^\alpha = \omega^\alpha(\mathbf{S}_\mathbf{X}, U^1, \dots, U^{m+s}) \quad , \quad \alpha = 1, \dots, m \quad , \quad (1.22)$$

$$U_T^\gamma = \langle Q^{\gamma 1} \rangle_{X^1} + \dots + \langle Q^{\gamma d} \rangle_{X^d} \quad , \quad \gamma = 1, \dots, m + s \quad ,$$

where $\langle Q^{\gamma p} \rangle = \langle Q^{\gamma p} \rangle(\mathbf{S}_\mathbf{X}, U^1, \dots, U^{m+s})$.

It can be shown then that the Hamiltonian structure of system (1.22) is given by the Poisson bracket

$$\begin{aligned}
\{S^\alpha(\mathbf{X}), S^\beta(\mathbf{Y})\} &= 0, \\
\{S^\alpha(\mathbf{X}), U^\gamma(\mathbf{Y})\} &= \omega^{\alpha\gamma}(\mathbf{S}_\mathbf{X}, U^1(\mathbf{X}), \dots, U^{m+s}(\mathbf{X})) \delta(\mathbf{X} - \mathbf{Y}), \\
\{U^\gamma(\mathbf{X}), U^\rho(\mathbf{Y})\} &= \langle A_{10\dots 0}^{\gamma\rho}(\mathbf{S}_\mathbf{X}, U^1(\mathbf{X}), \dots, U^{m+s}(\mathbf{X})) \rangle \delta_{X^1}(\mathbf{X} - \mathbf{Y}) + \\
&+ \dots + \langle A_{0\dots 01}^{\gamma\rho}(\mathbf{S}_\mathbf{X}, U^1(\mathbf{X}), \dots, U^{m+s}(\mathbf{X})) \rangle \delta_{X^d}(\mathbf{X} - \mathbf{Y}) + \\
&+ [\langle Q^{\gamma\rho p}(\mathbf{S}_\mathbf{X}, U^1(\mathbf{X}), \dots, U^{m+s}(\mathbf{X})) \rangle]_{X^p} \delta(\mathbf{X} - \mathbf{Y}), \quad \gamma, \rho = 1, \dots, m+s,
\end{aligned} \tag{1.23}$$

with the Hamiltonian functional

$$H_{av} = \int \langle P_H \rangle(\mathbf{S}_\mathbf{X}, U^1(\mathbf{X}), \dots, U^{m+s}(\mathbf{X})) d^d X$$

Let us note here, that although just a part of the integrals I^ν is formally used in the final construction of the Hamiltonian structure, the presence of the complete Hamiltonian set (I^1, \dots, I^N) plays an important role according to the scheme of [34]. The requirement of existence of the complete set of local conservation laws is actually rather strong in the multi-dimensional ($d > 1$) situation. Thus, for most of the integrable multi-dimensional systems the procedure, formulated above, can not be used for general $m > 1$ since only a finite set of local conservation laws is usually present in this case. On the other hand, the procedure usually works well in the single-phase ($m = 1$) case both in the integrable and non-integrable situations.

In this paper we are going to investigate the question if the necessary number of the integrals I^ν can be reduced still keeping the procedure of the bracket averaging well-justified. As we will show, the number of the integrals I^ν can be reduced in the case when a part of the phase variables θ_0 can in fact be represented as the pseudo-phases. As the analysis of different examples shows, this situation actually takes place quite often. Moreover, in many cases the existence of the multi-phase solutions for non-integrable systems is caused in fact by the presence of the pseudo-phases, playing the role of additional phases of the solutions. Thus, in many physical systems, the additional phases arise due to the presence of some global additional symmetries, corresponding to the additional integrals of the system. The multi-phase solutions can be considered in this case in fact as the periodic waves with the parameters $(k_1, \dots, k_d, \omega)$ in the nontrivial vacuum, while the additional parameters separate different vacua carrying permanent current. The simplest example of such situation can be given just by the nonlinear Shrödinger equation with d spatial dimensions, so we consider this example at the end of the paper.

In the next Chapter we will consider the procedure of the bracket averaging in the presence of the pseudo-phases.

2 The regularity conditions and the bracket averaging.

As we said in the previous Chapter, we will consider here systems (1.1) which can be represented in the Hamiltonian form with some local field-theoretic Poisson bracket (1.18) and the Hamiltonian

functional (1.19). Let us say, that the space of fields $\varphi(\mathbf{x})$ has a pseudo-phase structure with n_2 pseudo-phases if we have a (almost everywhere) free action of a n_2 -dimensional Abelian group G^{n_2} on the target space $(\varphi^1, \dots, \varphi^n)$. We will say here, that the pseudo-phase structure is compatible with the Poisson bracket (1.18) if the Poisson bracket is invariant under the action of G^{n_2} .

Easy to see that for the variables $(\varphi^1, \dots, \varphi^n)$, represented in the form (1.3), the compatibility of the pseudo-phase structure and the bracket (1.18) means that the functions $B_{(l_1, \dots, l_d)}^{ij}$ depend just on the spatial derivatives of the fields $(\phi^1(\mathbf{x}), \dots, \phi^{n_2}(\mathbf{x}))$. We can see, in particular, that the coefficients $B_{(l_1, \dots, l_d)}^{ij}$ represent quasiperiodic functions on the family Λ defined by formulas (1.4) - (1.5).

We will say also here that the Hamiltonian system (1.1) is compatible with the pseudo-phase structure if both the bracket (1.18) and the Hamiltonian functional (1.19) are invariant under the action of the group G^{n_2} .

According to the scheme of the previous Chapter, we are going to consider the Hamiltonian system (1.1), which is compatible with the action of the Abelian group

$$\begin{aligned} &(\rho^1(\mathbf{x}), \dots, \rho^{n_1}(\mathbf{x}), \phi^1(\mathbf{x}), \dots, \phi^{n_2}(\mathbf{x})) \rightarrow \\ &\rightarrow (\rho^1(\mathbf{x}), \dots, \rho^{n_1}(\mathbf{x}), \phi^1(\mathbf{x}) + \tau_0^1, \dots, \phi^{n_2}(\mathbf{x}) + \tau_0^{n_2}) \quad , \end{aligned} \quad (2.1)$$

and a complete regular family Λ of m -phase solutions of system (1.1) (or (1.7)) with n_2 pseudo-phases, represented by relations (1.4) - (1.5) and (1.6).

Let us define now a regular Hamiltonian family Λ of m -phase solutions with n_2 pseudo-phases.

Definition 2.1.

We call family Λ of m -phase solutions of system (1.1) (or (1.7)) with n_2 pseudo-phases a regular Hamiltonian family if :

- 1) *It represents a complete regular family of m -phase solutions of system (1.1) (or (1.7)) with n_2 pseudo-phases in the sense of Definition 1.1;*
- 2) *System (1.1) (or (1.7)) represents a Hamiltonian system compatible with the pseudo-phase structure given by the action of the group (2.1);*
- 3) *The Poisson bracket (1.18) has on Λ constant number of “annihilators” given by linearly independent quasiperiodic solutions $v_i^{(l)}(\mathbf{x})$ of the system*

$$\sum_{l_1, \dots, l_d} B_{(l_1, \dots, l_d)}^{ij}(\varphi, \varphi_{\mathbf{x}}, \dots) \Big|_{\Lambda} v_{j, l_1 x^1 \dots l_d x^d}^{(l)}(\mathbf{x}) = 0 \quad ,$$

where $v_i^{(l)}(\mathbf{x})$ have the same wave numbers $(\mathbf{k}_1, \dots, \mathbf{k}_d)$ as the corresponding functions $\varphi(\mathbf{x}) \in \Lambda$.

Let us consider now a set of the functionals I^ν , having the form

$$I^\nu = \int P^\nu(\rho, \rho_{\mathbf{x}}, \phi_{\mathbf{x}}, \rho_{\mathbf{xx}}, \phi_{\mathbf{xx}}, \dots) d^d x \quad (2.2)$$

Thus, we assume here that the functionals I^ν are invariant with respect to the action of the group G^{n_2} and the densities P^ν depend just on the derivatives of the fields ϕ . The functionals (2.2) can be considered on the space of rapidly decreasing functions just putting

$$I^\nu = \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} P^\nu(\rho, \rho_{\mathbf{x}}, \phi_{\mathbf{x}}, \rho_{\mathbf{xx}}, \phi_{\mathbf{xx}}, \dots) dx^1 \dots dx^d$$

or on the space of quasiperiodic functions, putting

$$I^\nu = \lim_{K \rightarrow \infty} \frac{1}{(2K)^d} \int_{-K}^K \dots \int_{-K}^K P^\nu(\boldsymbol{\rho}, \boldsymbol{\rho}_\mathbf{x}, \boldsymbol{\phi}_\mathbf{x}, \boldsymbol{\rho}_{\mathbf{x}\mathbf{x}}, \boldsymbol{\phi}_{\mathbf{x}\mathbf{x}}, \dots) dx^1 \dots dx^d$$

We will define also the variation derivatives of the functionals I^ν using the variations of $\boldsymbol{\rho}(\mathbf{x})$, $\boldsymbol{\phi}(\mathbf{x})$ with the same (rapidly decreasing or quasiperiodic) properties as the original functions. Easy to see then that in both cases just the standard Euler - Lagrange expressions for the variation derivatives can be used. It's not difficult to see also, that the functionals I^ν are also well-defined on the functions from the family Λ , having the form (1.4) - (1.5).

Let us assume everywhere below that the functionals (2.2) are defined in the appropriate way in accordance with the corresponding situation.

The pairwise Poisson brackets of the densities $P^\nu(\mathbf{x})$, $P^\mu(\mathbf{y})$ can be written in the form:

$$\{P^\nu(\mathbf{x}), P^\mu(\mathbf{y})\} = \sum_{l_1, \dots, l_d} A_{l_1 \dots l_d}^{\nu\mu}(\boldsymbol{\rho}, \boldsymbol{\rho}_\mathbf{x}, \boldsymbol{\phi}_\mathbf{x}, \dots) \delta^{(l_1)}(x^1 - y^1) \dots \delta^{(l_d)}(x^d - y^d)$$

where

$$A_{0 \dots 0}^{\nu\mu}(\boldsymbol{\rho}, \boldsymbol{\rho}_\mathbf{x}, \boldsymbol{\phi}_\mathbf{x}, \dots) \equiv \partial_{x^1} Q^{\nu\mu 1}(\boldsymbol{\rho}, \boldsymbol{\rho}_\mathbf{x}, \boldsymbol{\phi}_\mathbf{x}, \dots) + \dots + \partial_{x^d} Q^{\nu\mu d}(\boldsymbol{\rho}, \boldsymbol{\rho}_\mathbf{x}, \boldsymbol{\phi}_\mathbf{x}, \dots)$$

for some functions $Q^{\nu\mu q}(\boldsymbol{\rho}, \boldsymbol{\rho}_\mathbf{x}, \boldsymbol{\phi}_\mathbf{x}, \dots)$.

Definition 2.2.

We call a set (I^1, \dots, I^Q) , $Q = m(d+1) + n_2 + s$, of commuting functionals (2.2) a complete Hamiltonian set on a regular Hamiltonian family Λ of m -phase solutions of system (1.7) with n_2 pseudo-phases if:

1) The values of the functionals (I^1, \dots, I^Q) on any submanifold, given by the constraints

$$\mathbf{p}_1 = \text{const}, \dots, \mathbf{p}_d = \text{const},$$

in the space of parameters on Λ , give a complete set of parameters (U^1, \dots, U^Q) on this submanifold, excluding the initial phase shifts;

2) The Hamiltonian flows, generated by the functionals (I^1, \dots, I^Q) , generate on Λ linear phase shifts of $\boldsymbol{\theta}_0$ with frequencies $\boldsymbol{\omega}^\nu(\mathbf{U})$, and linear phase shifts of $\boldsymbol{\tau}_0$ with frequencies $\boldsymbol{\Omega}^\nu(\mathbf{U})$, such that

$$\text{rk} \begin{vmatrix} \omega^{\alpha\nu}(\mathbf{U}) \\ \Omega^{j\nu}(\mathbf{U}) \end{vmatrix} = m + n_2$$

3) At every "point" of the submanifold Λ the linear space generated by the variation derivatives $\delta I^\nu / \delta \varphi^i(\mathbf{x})$ contains the variation derivatives of all the corresponding annihilators of the bracket (1.18), such that we can write

$$v_i^{(l)}(\mathbf{x}, \mathbf{U}, \boldsymbol{\theta}_0) = \sum_{\nu=1}^Q \gamma_\nu^l(\mathbf{U}) \left. \frac{\delta I^\nu}{\delta \varphi^i(\mathbf{x})} \right|_\Lambda$$

for some functions $\gamma_\nu^l(\mathbf{U})$ on the family Λ .

We can see then that in the presence of a complete Hamiltonian set of the commuting functionals (2.2) the parameters \mathbf{U} on the family Λ can be also chosen in the form $(U^1, \dots, U^Q, \mathbf{p}_1, \dots, \mathbf{p}_d)$, where $U^\nu = \langle P^\nu \rangle$. We will also assume here that the Jacobian of the coordinate transformation

$$(\mathbf{k}_1, \dots, \mathbf{k}_d, \boldsymbol{\omega}, \mathbf{p}_1, \dots, \mathbf{p}_d, \boldsymbol{\Omega}, n^1, \dots, n^s) \rightarrow (U^1, \dots, U^N)$$

is different from zero whenever the values U^ν represent a complete set of parameters on Λ excluding the initial phase shifts $\boldsymbol{\theta}_0, \boldsymbol{\tau}_0$.

Let us consider now the functionals

$$J^\nu = \int_0^{2\pi} \dots \int_0^{2\pi} P^\nu \left(\boldsymbol{\rho}, k_1^{\beta_1} \boldsymbol{\rho}_{\theta^{\beta_1}}, \dots, k_d^{\beta_d} \boldsymbol{\rho}_{\theta^{\beta_d}}, k_1^{\gamma_1} \boldsymbol{\phi}_{\theta^{\gamma_1}} + \mathbf{p}_1, \dots \right) \frac{d^m \theta}{(2\pi)^m} \quad (2.3)$$

on the space of 2π -periodic in each θ^α functions $\boldsymbol{\rho}$ and $\boldsymbol{\phi}$.

The variation derivatives of the functionals J^ν

$$\zeta_{[\mathbf{U}]}^{(\nu)}(\boldsymbol{\theta} + \boldsymbol{\theta}_0) = \left(\left. \frac{\delta J^\nu}{\delta \rho^1(\boldsymbol{\theta})} \right|_\Lambda, \dots, \left. \frac{\delta J^\nu}{\delta \rho^{n_1}(\boldsymbol{\theta})} \right|_\Lambda, \left. \frac{\delta J^\nu}{\delta \phi^1(\boldsymbol{\theta})} \right|_\Lambda, \dots, \left. \frac{\delta J^\nu}{\delta \phi^{n_2}(\boldsymbol{\theta})} \right|_\Lambda \right) \quad (2.4)$$

represent left eigen-vectors of the operator $\hat{L}_{j[\mathbf{U}, \boldsymbol{\theta}_0]}^i$ with zero eigenvalues, which depend regularly on parameters \mathbf{U} on Λ . Since the number of independent parameters $(\mathbf{k}_1, \dots, \mathbf{k}_d)$ on Λ is equal to md , we can claim that the number of linear independent vectors (2.4) should not be less than $m + n_2 + s$ for a complete Hamiltonian set of the functionals I^ν on Λ according to the first requirement of Definition 2.2. Thus, we can formulate here the following Proposition:

Proposition 2.1.

Let the set of the functionals (I^1, \dots, I^Q) represent a complete Hamiltonian set on a regular Hamiltonian family of m -phase solutions of (1.7) Λ with n_2 pseudo-phases. Then the linear span of the vectors (2.4) contains all the regular left eigen-vectors $\boldsymbol{\kappa}_{[\mathbf{U}]}^{(q)}(\boldsymbol{\theta} + \boldsymbol{\theta}_0)$ of the operator $\hat{L}_{j[\mathbf{U}, \boldsymbol{\theta}_0]}^i$ with zero eigen-values.

From the other hand, for a complete Hamiltonian family Λ we can then claim also, that the number of the linearly independent vectors (2.4) is exactly equal to the number of $\boldsymbol{\kappa}_{[\mathbf{U}]}^{(q)}(\boldsymbol{\theta} + \boldsymbol{\theta}_0)$ representing all the linearly independent regular left eigen-vectors of the operator $\hat{L}_{j[\mathbf{U}, \boldsymbol{\theta}_0]}^i$ with zero eigen-values. As a corollary, we can formulate here the following Lemma, which will be rather important in our further considerations.

Lemma 2.1.

Let the set of the functionals (I^1, \dots, I^Q) represent a complete Hamiltonian set on a regular Hamiltonian family Λ of m -phase solutions of (1.7) with n_2 pseudo-phases. Consider the corresponding functions

$$k_p^\alpha = k_p^\alpha(U^1, \dots, U^Q, \mathbf{p}_1, \dots, \mathbf{p}_d)$$

in the coordinate system $(U^1, \dots, U^Q, \mathbf{p}_1, \dots, \mathbf{p}_d)$. Then the functionals

$$k_p^\alpha(J^1, \dots, J^Q, \mathbf{p}_1, \dots, \mathbf{p}_d)$$

have identically zero variation derivatives w.r.t. $\boldsymbol{\rho}$ and $\boldsymbol{\phi}$ on Λ .

Proof.

Indeed, the conditions of the Lemma imply that the number of the linearly independent vectors $\zeta_{[\mathbf{U}]}^{(\nu)}(\boldsymbol{\theta})$ on Λ is equal to $m + n_2 + s$. As a corollary, we can write md independent relations

$$\sum_{\nu=1}^Q \lambda_{\nu}^{\tau}(\mathbf{U}) \zeta_{[\mathbf{U}]}^{(\nu)}(\boldsymbol{\theta} + \boldsymbol{\theta}_0) \equiv 0 \quad , \quad \tau = 1, \dots, md$$

with some functions $\lambda_{\nu}^{\tau}(\mathbf{U})$ on Λ .

For the corresponding coordinates U^{ν} on Λ we can write then the relations

$$\sum_{\nu=1}^Q \lambda_{\nu}^{\tau}(\mathbf{U}) dU^{\nu} = \sum_{q=1}^d \sum_{\beta=1}^m \mu_{(\beta q)}^{(\tau)}(\mathbf{U}) dk_q^{\beta}(\mathbf{U})$$

with some matrix $\mu_{(\beta q)}^{(\tau)}(\mathbf{U})$.

Since the values $\mathbf{U} = (U^1, \dots, U^Q, \mathbf{p}_1, \dots, \mathbf{p}_d)$ represent a coordinate system on Λ , the matrix $\mu_{(\beta q)}^{(\tau)}(\mathbf{U})$ is invertible and we can write the relations

$$dk_q^{\beta} = \sum_{\tau=1}^{md} (\hat{\mu}^{-1})_{(\tau)}^{(\beta q)}(\mathbf{U}) \sum_{\nu=1}^N \lambda_{\nu}^{(\tau)}(\mathbf{U}) dU^{\nu}$$

for every k_q^{β} , which gives the proof of the Lemma.

Lemma 2.1 is proved.

As a corollary of Lemma 2.1 we can claim that the functionals $k_p^{\alpha}(I^1, \dots, I^Q, \mathbf{p}_1, \dots, \mathbf{p}_d)$ generate zero flows on the family Λ . Using Definition 2.2 we can write then

$$\sum_{\nu=1}^Q \frac{\partial k_p^{\alpha}(U^1, \dots, U^Q, \mathbf{p}_1, \dots, \mathbf{p}_d)}{\partial U^{\nu}} \omega^{\beta \nu}(U^1, \dots, U^Q, \mathbf{p}_1, \dots, \mathbf{p}_d) \equiv 0 \quad (2.5)$$

$$\sum_{\nu=1}^Q \frac{\partial k_p^{\alpha}(U^1, \dots, U^Q, \mathbf{p}_1, \dots, \mathbf{p}_d)}{\partial U^{\nu}} \Omega^{j \nu}(U^1, \dots, U^Q, \mathbf{p}_1, \dots, \mathbf{p}_d) \equiv 0 \quad (2.6)$$

for the functions $k_p^{\alpha}(U^1, \dots, U^Q, \mathbf{p}_1, \dots, \mathbf{p}_d)$ on Λ .

Finally, let us note also, that in the presence of a complete Hamiltonian set (I^1, \dots, I^Q) for a regular Hamiltonian family Λ of m -phase solutions of (1.7) with n_2 pseudo-phases we can claim in fact, that the number of annihilators of the bracket (1.18) on Λ is equal to the number of the additional parameters (n^1, \dots, n^s) . Indeed, according to the requirements (2)-(3) of Definition 2.2, the number of the linearly independent vectors (2.4) is equal to $m + n_2 + s$, where s is the number of annihilators of the bracket (1.18) on Λ . Comparing this number with the number of the vectors $\kappa_{[\mathbf{U}]}^{(q)}(\boldsymbol{\theta} + \boldsymbol{\theta}_0)$ we get the required statement.

Let us discuss now the procedure of the bracket averaging. Our considerations here will follow in many features the scheme of [33, 34].

Let us introduce the extended field space $\varphi(\mathbf{x}) \rightarrow \varphi(\boldsymbol{\theta}, \mathbf{X})$, where all the functions $\varphi(\boldsymbol{\theta}, \mathbf{X})$ are 2π -periodic in each θ^{α} , and consider the Poisson bracket

$$\{\varphi^i(\boldsymbol{\theta}, \mathbf{X}), \varphi^j(\boldsymbol{\theta}', \mathbf{Y})\} = \sum_{l_1, \dots, l_d} \epsilon^{l_1 + \dots + l_d} B_{(l_1, \dots, l_d)}^{ij}(\varphi, \epsilon \varphi_{\mathbf{X}}, \dots) \delta_{l_1 X^1 \dots l_d X^d}(\mathbf{X} - \mathbf{Y}) \delta(\boldsymbol{\theta} - \boldsymbol{\theta}') \quad (2.7)$$

on the space of fields $\varphi(\boldsymbol{\theta}, \mathbf{X})$.

For convenience we will define here the delta-function $\delta(\boldsymbol{\theta} - \boldsymbol{\theta}')$ and its higher derivatives $\delta_{\theta^{\alpha_1} \dots \theta^{\alpha_s}}(\boldsymbol{\theta} - \boldsymbol{\theta}')$ on the space of 2π -periodic functions by the formula

$$\int_0^{2\pi} \dots \int_0^{2\pi} \delta_{\theta^{\alpha_1} \dots \theta^{\alpha_s}}(\boldsymbol{\theta} - \boldsymbol{\theta}') \psi(\boldsymbol{\theta}') \frac{d^m \theta'}{(2\pi)^m} \equiv \psi_{\theta^{\alpha_1} \dots \theta^{\alpha_s}}(\boldsymbol{\theta})$$

Also we put here the rule

$$\delta S \equiv \int_0^{2\pi} \dots \int_0^{2\pi} \frac{\delta S}{\delta \varphi^i(\boldsymbol{\theta})} \delta \varphi^i(\boldsymbol{\theta}) \frac{d^m \theta}{(2\pi)^m}$$

in the definition of the corresponding variation derivatives.

Consider now the submanifold \mathcal{K} in the extended field space defined by the following conditions:

1) For given functions $\{\mathbf{S}(\mathbf{X}), \boldsymbol{\Sigma}(\mathbf{X}), \mathbf{U}(\mathbf{X})\}$ the functions $\varphi(\boldsymbol{\theta}, \mathbf{X}) \in \mathcal{K}$ are defined by the formulas

$$\begin{aligned} \rho^i(\boldsymbol{\theta}, \mathbf{X}) &= R^i \left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}, \mathbf{U}(\mathbf{X}) \right), \quad i = 1, \dots, n_1, \\ \phi^j(\boldsymbol{\theta}, \mathbf{X}) &= \Psi^j \left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}, \mathbf{U}(\mathbf{X}) \right) + \frac{1}{\epsilon} \Sigma^j(\mathbf{X}), \quad j = 1, \dots, n_2 \end{aligned} \quad (2.8)$$

where the values \mathbf{U} represent the full set of parameters on Λ excluding the initial phase shifts.

2) The functions $\mathbf{U}(\mathbf{X})$ are connected with the functions $\mathbf{S}(\mathbf{X})$ and $\boldsymbol{\Sigma}(\mathbf{X})$ by the relations

$$k_q^\alpha(\mathbf{U}(\mathbf{X})) = S_{X^q}^\alpha, \quad p_q^j(\mathbf{U}(\mathbf{X})) = \Sigma_{X^q}^j \quad (2.9)$$

where \mathbf{k}_q and \mathbf{p}_q are the corresponding wave numbers and “pseudo wave numbers” defined on the family Λ .

Thus, the elements $\varphi(\boldsymbol{\theta}, \mathbf{X}) \in \mathcal{K}$ are parametrized by the functions $\{\mathbf{S}(\mathbf{X}), \boldsymbol{\Sigma}(\mathbf{X}), \mathbf{U}(\mathbf{X})\}$ with relations (2.9) and are connected with the zero approximation (1.10) for the modulated m -phase solutions of (1.7). In the presence of a complete Hamiltonian set of integrals I^ν the parameters \mathbf{U} can be chosen in the form $(U^1, \dots, U^Q, \mathbf{p}_1, \dots, \mathbf{p}_d)$, where $U^\nu \equiv \langle P^\nu \rangle$ and \mathbf{p}_q are given by relations (2.9).

Let us introduce the functionals

$$\Sigma^i(\mathbf{X}) = \epsilon \int_0^{2\pi} \dots \int_0^{2\pi} \phi^j(\boldsymbol{\theta}, \mathbf{X}) \frac{d^m \theta}{(2\pi)^m} \quad (2.10)$$

It is easy to see that the values of $\Sigma^i(\mathbf{X})$ on the functions $\varphi(\boldsymbol{\theta}, \mathbf{X}) \in \mathcal{K}$ coincide with the corresponding parameters on \mathcal{K} . We can consider then the parameters $\boldsymbol{\Sigma}(\mathbf{X})$ and $(\mathbf{p}_1(\mathbf{X}), \dots, \mathbf{p}_d(\mathbf{X}))$ as the functionals on the whole extended field space, having the appropriate values on the submanifold \mathcal{K} .

To introduce the analogous functionals for the parameters $U^\nu(\mathbf{X})$ let us introduce the functionals

$$J^\nu(\mathbf{X}) = \int_0^{2\pi} \dots \int_0^{2\pi} P^\nu(\boldsymbol{\rho}, \epsilon \boldsymbol{\rho}_\mathbf{X}, \epsilon \boldsymbol{\phi}_\mathbf{X}, \dots) \frac{d^m \theta}{(2\pi)^m}, \quad \nu = 1, \dots, Q,$$

and consider their values on the submanifold \mathcal{K} .

Easy to see that we can write on \mathcal{K} :

$$J^\nu(\mathbf{X}) = U^\nu(\mathbf{X}) + \sum_{l \geq 1} \epsilon^l J_{(l)}^\nu(\mathbf{X}) \quad , \quad \nu = 1, \dots, Q \quad (2.11)$$

where $J_{(l)}^\nu$ are some local functions of $(U^1(\mathbf{X}), \dots, U^Q(\mathbf{X}), \mathbf{p}_1(\mathbf{X}), \dots, \mathbf{p}_d(\mathbf{X}))$ and their spatial derivatives which are polynomial in the derivatives and have grading degree l in terms of the total number of differentiations with respect to \mathbf{X} .

Let us say that the higher terms in (2.11) are in fact not uniquely defined on \mathcal{K} due to the compatibility relations (1.14). It is in fact sufficient for us that the terms $J_{(l)}^\nu(\mathbf{X})$ can be chosen in some definite way in every order $l \geq 1$. Let us note also that the corresponding choice affects the definition of the functionals $U^\nu(\mathbf{X})$ just in the higher orders in ϵ ($l \geq 1$) which is actually not important for the construction.

Transformation (2.11) can be also inverted as a formal series in ϵ , such that we have

$$U^\nu(\mathbf{X}) = J^\nu(\mathbf{X}) + \sum_{l \geq 1} \epsilon^l U_{(l)}^\nu(\mathbf{X}) \quad , \quad \nu = 1, \dots, Q \quad (2.12)$$

on the submanifold \mathcal{K} . Now the functions $U_{(l)}^\nu$ represent local functions of $(J^1(\mathbf{X}), \dots, J^Q(\mathbf{X}), \mathbf{p}_1(\mathbf{X}), \dots, \mathbf{p}_d(\mathbf{X}))$ and their spatial derivatives, polynomial in the derivatives, and having degree l in terms of the total number of differentiations w.r.t. \mathbf{X} . Now, we can consider the values $U^\nu(\mathbf{X})$ as the functionals on the whole extended field space.

Let us put for simplicity the boundary conditions $k_1^\alpha(X^1, 0, \dots, 0) \rightarrow 0$, $X^1 \rightarrow -\infty$, for the functionals $k_1^\alpha(\mathbf{U}, \Sigma_{\mathbf{X}})$ on the extended functional space and define also the functionals $S^\alpha(\mathbf{X})$ by the formula

$$S^\alpha(\mathbf{X}) = \int_{-\infty}^{X^1} k_1^\alpha(X'^1, 0, \dots, 0) dX'^1 + \dots + \int_0^{X^d} k_d^\alpha(X^1, \dots, X^{d-1}, X'^d) dX'^d \quad (2.13)$$

Now, all the parameters on the submanifold \mathcal{K} are defined as functionals on the whole extended field space. Let us note that on the submanifold \mathcal{K} we naturally have the relations $S_{X^q}^\alpha = k_q^\alpha(\mathbf{X})$ which are in general not true outside \mathcal{K} .

Let us consider now the Poisson brackets of the functionals, introduced above, on the submanifold \mathcal{K} . According to the definition of the functionals $\Sigma(\mathbf{X})$ and Definition 2.2 it is not difficult to get the following relations for the brackets of $\Sigma(\mathbf{X})$ and $U^\mu(\mathbf{Y})$ on \mathcal{K} :

$$\{\Sigma^j(\mathbf{X}), \Sigma^l(\mathbf{Y})\}|_{\mathcal{K}} = O(\epsilon^2) \quad , \quad (2.14)$$

$$\{\Sigma^j(\mathbf{X}), U^\mu(\mathbf{Y})\}|_{\mathcal{K}} = \epsilon \Omega^{j\mu}(\mathbf{X}) \delta(\mathbf{X} - \mathbf{Y}) + O(\epsilon^2) \quad , \quad (2.15)$$

$j, l = 1, \dots, n_2$, $\nu = 1, \dots, Q$.

Using relations (2.6) and (2.14) - (2.15) we can then write

$$\{\Sigma^j(\mathbf{X}), k_p^\alpha(\mathbf{Y})\}|_{\mathcal{K}} = O(\epsilon^2) \quad (2.16)$$

for the functionals $k_p^\alpha(U^1(\mathbf{X}), \dots, U^Q(\mathbf{X}), \Sigma_{X^1}, \dots, \Sigma_{X^d})$.

The pairwise Poisson brackets of the functionals $U^\nu(\mathbf{X})$ have the order $O(\epsilon)$ everywhere on the extended field space and we can write on \mathcal{K} :

$$\begin{aligned} \{U^\nu(\mathbf{X}), U^\mu(\mathbf{Y})\}|_{\mathcal{K}} &= \{J^\nu(\mathbf{X}), J^\mu(\mathbf{Y})\}|_{\mathcal{K}} + O(\epsilon^2) = \\ &= \epsilon \langle A_{10\dots 0}^{\nu\mu} \rangle(\mathbf{X}) \delta_{X^1}(\mathbf{X} - \mathbf{Y}) + \dots + \epsilon \langle A_{0\dots 01}^{\nu\mu} \rangle(\mathbf{X}) \delta_{X^d}(\mathbf{X} - \mathbf{Y}) + \\ &+ \epsilon [\langle Q^{\nu\mu p} \rangle]_{X^p} \delta(\mathbf{X} - \mathbf{Y}) + O(\epsilon^2) \quad , \quad \nu, \mu = 1, \dots, Q \end{aligned}$$

Let us prove here the following important Lemma:

Lemma 2.2.

Let (I^1, \dots, I^Q) , $Q = m(d+1) + n_2 + s$, represent a complete Hamiltonian set of commuting functionals on a regular Hamiltonian family Λ of m -phase solutions of (1.7) with n_2 pseudo-phases. Consider the corresponding functions $k_p^\alpha(\mathbf{U}) = k_p^\alpha(U^1, \dots, U^Q, \mathbf{p}_1, \dots, \mathbf{p}_d)$ on the family Λ .

Consider any functional \tilde{I} of the form

$$\tilde{I} = \int \tilde{P}(\boldsymbol{\rho}, \boldsymbol{\rho}_{\mathbf{x}}, \boldsymbol{\phi}_{\mathbf{x}}, \boldsymbol{\rho}_{\mathbf{x}\mathbf{x}}, \boldsymbol{\phi}_{\mathbf{x}\mathbf{x}}, \dots) d^d x \quad , \quad (2.17)$$

leaving the family Λ and the parameters \mathbf{U} invariant and generating on Λ linear shifts of θ_0^α with frequencies $\tilde{\omega}^\alpha(\mathbf{U})$ and linear shifts of τ_0^j with frequencies $\tilde{\Omega}^j(\mathbf{U})$. Let us consider the functionals

$$\tilde{J}(\mathbf{X}) = \int_0^{2\pi} \dots \int_0^{2\pi} \tilde{P}(\boldsymbol{\rho}, \epsilon \boldsymbol{\rho}_{\mathbf{x}}, \epsilon \boldsymbol{\phi}_{\mathbf{x}}, \epsilon^2 \boldsymbol{\rho}_{\mathbf{x}\mathbf{x}}, \epsilon^2 \boldsymbol{\phi}_{\mathbf{x}\mathbf{x}}, \dots) \frac{d^m \theta}{(2\pi)^m}$$

Then the functionals $k_p^\alpha(U^1(\mathbf{X}), \dots, U^Q(\mathbf{X}), \boldsymbol{\Sigma}_{X^1}, \dots, \boldsymbol{\Sigma}_{X^d})$ have the following Poisson brackets with the functionals $\tilde{J}(\mathbf{Y})$ on \mathcal{K} :

$$\left\{ k_p^\alpha(\mathbf{X}), \tilde{J}(\mathbf{Y}) \right\} \Big|_{\mathcal{K}} = \epsilon [\tilde{\omega}^\alpha(\mathbf{X}) \delta(\mathbf{X} - \mathbf{Y})]_{X^p} + O(\epsilon^2)$$

Proof.

Consider the dynamical system generated by the functional

$$\tilde{J}_{[q]} = \int \tilde{J}(\mathbf{Y}) q(\mathbf{Y}) d^d Y \quad (2.18)$$

with compactly supported $q(\mathbf{Y})$ according to bracket (2.7).

It is easy to see that in the main order ($O(1)$) the corresponding evolution leaves invariant the submanifold \mathcal{K} , generating the shifts of the functions $\mathbf{S}(\mathbf{X})$ and $\boldsymbol{\Sigma}(\mathbf{X})$ with the frequencies $\epsilon q(\mathbf{X}) \tilde{\omega}(\mathbf{X})$ and $\epsilon q(\mathbf{X}) \tilde{\Omega}(\mathbf{X})$ respectively. As a result, we can decompose the dynamical system on \mathcal{K} into two parts:

1) The dynamics along the submanifold \mathcal{K} giving the shifts of parameters $\mathbf{S}(\mathbf{X})$ and $\boldsymbol{\Sigma}(\mathbf{X})$ with the frequencies $\epsilon q(\mathbf{X}) \tilde{\omega}(\mathbf{X})$ and $\epsilon q(\mathbf{X}) \tilde{\Omega}(\mathbf{X})$;

2) The additional dynamics of the order $O(\epsilon)$ having the form

$$\varphi_t^i = \epsilon \tilde{\eta}_{[q]}^i \left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}, \mathbf{X} \right)$$

with some 2π -periodic in each θ^α functions $\tilde{\eta}_{[q]}^i(\boldsymbol{\theta}, \mathbf{X})$ on \mathcal{K} .

The first part gives the following evolution of $k_p^\alpha(\mathbf{X})$ on \mathcal{K} :

$$k_{pt}^\alpha = \epsilon (q(\mathbf{X}) \tilde{\omega}^\alpha(\mathbf{X}))_{X^p}$$

according to the definition of the functionals $k_p^\alpha(\mathbf{X})$ on \mathcal{K} .

To get the contribution of the second part to the evolution of $k_p^\alpha(\mathbf{X})$ we can change in the main part the functionals $k_p^\alpha(U^1(\mathbf{X}), \dots, U^Q(\mathbf{X}), \Sigma_{X^1}, \dots, \Sigma_{X^d})$ to $k_p^\alpha(J^1(\mathbf{X}), \dots, J^Q(\mathbf{X}), \Sigma_{X^1}, \dots, \Sigma_{X^d})$ using (2.11) - (2.12). It's not difficult to see then that the main contribution of the corresponding dynamics to the evolution of $k_p^\alpha(\mathbf{X})$ is given by the convolution of the variation derivatives of the corresponding functionals $k_p^\alpha(J^1, \dots, J^Q, \mathbf{p}_1, \dots, \mathbf{p}_d)$, defined on the space of 2π -periodic in each θ^α functions $(\boldsymbol{\rho}(\boldsymbol{\theta}), \boldsymbol{\phi}(\boldsymbol{\theta}))$, with the functions $\tilde{\eta}_{[q]}^i(\boldsymbol{\theta}, \mathbf{X})$ at every given \mathbf{X} . According to Lemma 2.1 we get then that the corresponding contribution is absent in the order $O(\epsilon)$.

Finally, we can write on \mathcal{K} :

$$\left\{ k_p^\alpha(\mathbf{X}), \tilde{J}_{[q]} \right\} \Big|_{\mathcal{K}} = \epsilon (q(\mathbf{X}) \tilde{\omega}^\alpha(\mathbf{X}))_{X^p}$$

which is equivalent to the assertion of the Lemma.

Lemma 2.2 is proved.

As a corollary from Lemma 2.2 we can write, in particular

$$\left\{ k_p^\alpha(\mathbf{X}), U^\mu(\mathbf{Y}) \right\} \Big|_{\mathcal{K}} = \epsilon [\omega^{\alpha\mu}(\mathbf{X}) \delta(\mathbf{X} - \mathbf{Y})]_{X^p} + O(\epsilon^2) \quad (2.19)$$

for the functionals $U^\mu(\mathbf{Y})$, using the analogous relations for $J^\mu(\mathbf{Y})$ and relations (2.12).

From relations (2.5), (2.16) and (2.19) it is not difficult to get then also the following relations on \mathcal{K} :

$$\left\{ k_p^\alpha(\mathbf{X}), k_q^\beta(\mathbf{Y}) \right\} \Big|_{\mathcal{K}} = O(\epsilon^2) \quad (2.20)$$

Using the definition (2.13) of the functionals $S^\alpha(\mathbf{X})$ and relations (2.16), (2.19), (2.20), we can write then the following relations for their Poisson brackets on \mathcal{K} :

$$\begin{aligned} \left\{ S^\alpha(\mathbf{X}), \Sigma^j(\mathbf{Y}) \right\} \Big|_{\mathcal{K}} &= O(\epsilon^2) \quad , \quad \left\{ S^\alpha(\mathbf{X}), S^\beta(\mathbf{Y}) \right\} \Big|_{\mathcal{K}} = O(\epsilon^2) \\ \left\{ S^\alpha(\mathbf{X}), U^\mu(\mathbf{Y}) \right\} \Big|_{\mathcal{K}} &= \epsilon \omega^{\alpha\mu}(\mathbf{X}) \delta(\mathbf{X} - \mathbf{Y}) + O(\epsilon^2) \end{aligned} \quad (2.21)$$

It will be convenient now to choose the parameters on the family Λ in the form

$$(\mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{p}_1, \dots, \mathbf{p}_d, U^1, \dots, U^{m+n_2+s}) \quad (2.22)$$

where $U^\gamma \equiv \langle P^\gamma \rangle$, $\gamma = 1, \dots, m + n_2 + s$, represent just a subset of the set U^ν , $\nu = 1, \dots, Q = m(d+1) + n_2 + s$, and to consider the functionals

$$\{\mathbf{S}(\mathbf{X}), \Sigma(\mathbf{X}), U^1(\mathbf{X}), \dots, U^{m+n_2+s}(\mathbf{X})\}$$

as completely independent ‘‘coordinates’’ on \mathcal{K} according to (2.9). Let us say that the subset $\{U^\gamma\}$ can be chosen in arbitrary way just to give a functionally independent system (2.22). For convenience, we will denote now by \mathbf{U} just a set of the functionals U^γ : $\mathbf{U} = (U^1, \dots, U^{m+n_2+s})$.

Let us introduce also the “constraints” $g^i(\boldsymbol{\theta}, \mathbf{X})$ near \mathcal{K} just putting in general form:

$$g^i(\boldsymbol{\theta}, \mathbf{X}) = \varphi^i(\boldsymbol{\theta}, \mathbf{X}) - \Phi^i \left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}, \mathbf{S}_{\mathbf{X}}, \boldsymbol{\Sigma}_{\mathbf{X}}, \mathbf{U}(\mathbf{X}), \boldsymbol{\Sigma}(\mathbf{X}) \right)$$

where Φ^i represent the right-hand part of relations (2.8). We have to note that the functionals $g^i(\boldsymbol{\theta}, \mathbf{X})$ are not independent. Thus, the following relations for the “gradients” of $g^i(\boldsymbol{\theta}, \mathbf{X})$ can be written on \mathcal{K} :

$$\int \int_0^{2\pi} \dots \int_0^{2\pi} \frac{\delta G(\mathbf{Z})}{\delta \varphi^i(\boldsymbol{\theta}, \mathbf{X})} \Big|_{\mathcal{K}} \frac{\delta g^i(\boldsymbol{\theta}, \mathbf{X})}{\delta \varphi^j(\boldsymbol{\theta}', \mathbf{Y})} \Big|_{\mathcal{K}} \frac{d^m \theta}{(2\pi)^m} d^d X \equiv 0 \quad (2.23)$$

where $G(\mathbf{Z})$ represents any of the functionals $S^\alpha(\mathbf{Z})$, $\boldsymbol{\Sigma}^j(\mathbf{Z})$ or $U^\gamma(\mathbf{Z})$.

Using Lemma 2.2 we can write also the relations

$$\left\{ g^i(\boldsymbol{\theta}, \mathbf{X}), \tilde{J}_{[q]} \right\} \Big|_{\mathcal{K}} = O(\epsilon)$$

for any functional $\tilde{J}_{[q]}$ defined by (2.18) with $\tilde{J}(\mathbf{Y})$ satisfying the requirements of Lemma 2.2. In particular, for the functionals

$$J_{[q]} = \int J^\mu(\mathbf{Y}) q_\mu(\mathbf{Y}) d^d Y, \quad U_{[q]} = \int U^\mu(\mathbf{Y}) q_\mu(\mathbf{Y}) d^d Y$$

with compactly supported $q_\mu(\mathbf{Y})$, $\mu = 1, \dots, Q$, we can write

$$\left\{ g^i(\boldsymbol{\theta}, \mathbf{X}), J_{[q]} \right\} \Big|_{\mathcal{K}} = O(\epsilon), \quad \left\{ g^i(\boldsymbol{\theta}, \mathbf{X}), U_{[q]} \right\} \Big|_{\mathcal{K}} = O(\epsilon) \quad (2.24)$$

Using the definition of the functionals $\boldsymbol{\Sigma}^j(\mathbf{X})$ and relations (2.21) we can write the same relations also for the functionals $\boldsymbol{\Sigma}_{[p]} = \int \boldsymbol{\Sigma}^j(\mathbf{Y}) p_j(\mathbf{Y}) d^d Y$, i.e.

$$\left\{ g^i(\boldsymbol{\theta}, \mathbf{X}), \boldsymbol{\Sigma}_{[p]} \right\} \Big|_{\mathcal{K}} = O(\epsilon) \quad (2.25)$$

We will need now another important Lemma:

Lemma 2.3.

Let Λ be a complete regular family of m -phase solutions of system (1.7) with n_2 pseudo-phases and system (1.12) - (1.14) represent the corresponding regular Whitham system on Λ . Let system (1.7) has the first integral \tilde{I} of the form (2.17) such that we have

$$\tilde{P}_t(\boldsymbol{\rho}, \boldsymbol{\rho}_{\mathbf{x}}, \boldsymbol{\phi}_{\mathbf{x}}, \dots) = \tilde{Q}_{x^1}^1(\boldsymbol{\rho}, \boldsymbol{\rho}_{\mathbf{x}}, \boldsymbol{\phi}_{\mathbf{x}}, \dots) + \dots + \tilde{Q}_{x^d}^d(\boldsymbol{\rho}, \boldsymbol{\rho}_{\mathbf{x}}, \boldsymbol{\phi}_{\mathbf{x}}, \dots)$$

on the solutions of (1.7). Then the Whitham system (1.12) - (1.14) implies the relation

$$\langle \tilde{P} \rangle_T = \langle \tilde{Q}^1 \rangle_{X^1} + \dots + \langle \tilde{Q}^d \rangle_{X^d}.$$

Proof.

Easy to see that for any time dependence of the parameters on the family Λ we can write:

$$\langle \tilde{P} \rangle_T = \int_0^{2\pi} \dots \int_0^{2\pi} \left(\frac{\delta \tilde{J}}{\delta \rho^i(\boldsymbol{\theta})} \Big|_{\Lambda} R_T^i(\boldsymbol{\theta}) + \frac{\delta \tilde{J}}{\delta \phi^j(\boldsymbol{\theta})} \Big|_{\Lambda} \Psi_T^j(\boldsymbol{\theta}) \right) \frac{d^m \theta}{(2\pi)^m} +$$

$$+ k_{qT}^\beta \frac{\partial \tilde{J}}{\partial k_q^\beta} \Big|_\Lambda + p_{qT}^j \frac{\partial \tilde{J}}{\partial p_q^j} \Big|_\Lambda$$

where the functional

$$\tilde{J} = \int_0^{2\pi} \dots \int_0^{2\pi} \tilde{P} \left(\boldsymbol{\rho}, k_1^{\beta_1} \boldsymbol{\rho}_{\theta^{\beta_1}}, \dots, k_d^{\beta_d} \boldsymbol{\rho}_{\theta^{\beta_d}}, k_1^{\gamma_1} \boldsymbol{\phi}_{\theta^{\gamma_1}} + \mathbf{p}_1, \dots \right) \frac{d^m \theta}{(2\pi)^m}$$

is defined on the space of 2π -periodic functions for any given parameters $(\mathbf{k}_1, \dots, \mathbf{k}_d)$, $(\mathbf{p}_1, \dots, \mathbf{p}_d)$.

Let us also introduce the functions

$$\tilde{\Pi}_{\boldsymbol{\rho}^i}^{(l_1 \dots l_d)}(\boldsymbol{\rho}, \boldsymbol{\rho}_\mathbf{x}, \boldsymbol{\phi}_\mathbf{x}, \dots) \equiv \frac{\partial \tilde{P}(\boldsymbol{\rho}, \boldsymbol{\rho}_\mathbf{x}, \boldsymbol{\phi}_\mathbf{x}, \dots)}{\partial \rho_{l_1 x^1 \dots l_d x^d}^i},$$

$$\tilde{\Pi}_{\boldsymbol{\phi}^j}^{(l_1 \dots l_d)}(\boldsymbol{\rho}, \boldsymbol{\rho}_\mathbf{x}, \boldsymbol{\phi}_\mathbf{x}, \dots) \equiv \frac{\partial \tilde{P}(\boldsymbol{\rho}, \boldsymbol{\rho}_\mathbf{x}, \boldsymbol{\phi}_\mathbf{x}, \dots)}{\partial \phi_{l_1 x^1 \dots l_d x^d}^j}, \quad l_1, \dots, l_d \geq 0$$

We can then write according to (1.7)

$$\epsilon \tilde{Q}_{X^1}^1 + \dots + \epsilon \tilde{Q}_{X^d}^d \equiv \sum_{l_1, \dots, l_d} \epsilon^{l_1 + \dots + l_d} \left(\tilde{\Pi}_{\boldsymbol{\rho}^i}^{(l_1 \dots l_d)} A_{l_1 x^1 \dots l_d x^d}^i + \tilde{\Pi}_{\boldsymbol{\phi}^j}^{(l_1 \dots l_d)} B_{l_1 x^1 \dots l_d x^d}^j \right)$$

The fulfillment of conditions (1.14) permits us to introduce the functions $\mathbf{S}(\mathbf{X})$, $\boldsymbol{\Sigma}(\mathbf{X})$ and consider the functions $\tilde{P}(\boldsymbol{\rho}, \boldsymbol{\rho}_\mathbf{x}, \boldsymbol{\phi}_\mathbf{x}, \dots)$ and $\tilde{Q}(\boldsymbol{\rho}, \boldsymbol{\rho}_\mathbf{x}, \boldsymbol{\phi}_\mathbf{x}, \dots)$ on the submanifold \mathcal{K} .

Easy to see that the operators $\epsilon \partial / \partial X^p$ on the submanifold \mathcal{K} can be naturally represented as a sum of $k_p^\alpha \partial / \partial \theta^\alpha$ and the terms proportional to ϵ . So, let us introduce on \mathcal{K} the natural expansion for any expression $f(\boldsymbol{\rho}, \boldsymbol{\rho}_\mathbf{x}, \boldsymbol{\phi}_\mathbf{x}, \dots)$ invariant under the action of the pseudo-phase group:

$$f(\boldsymbol{\rho}, \boldsymbol{\rho}_\mathbf{x}, \boldsymbol{\phi}_\mathbf{x}, \dots) \Big|_{\mathcal{K}} = \sum_{l \geq 0} \epsilon^l f_{[l]} \left[\frac{\mathbf{S}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}; \mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{p}_1, \dots, \mathbf{p}_d, \mathbf{U} \right]$$

where $f_{[l]}$ are smooth functions of the arguments $(\mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{p}_1, \dots, \mathbf{p}_d, \mathbf{U})$ and their \mathbf{X} -derivatives, polynomial in the derivatives and having degree l in terms of the total number of differentiations of these parameters w.r.t. \mathbf{X} . Since the common phase shift is not important in the integration w.r.t. $\boldsymbol{\theta}$, let us also assume below that the phase shift $\mathbf{S}(\mathbf{X})/\epsilon$ is omitted in the functions $f_{[l]}$ after taking all the differentiations w.r.t. \mathbf{X} .

We can write then (summation over all the repeated indexes):

$$\begin{aligned} \langle \tilde{Q}^1 \rangle_{X^1} + \dots + \langle \tilde{Q}^d \rangle_{X^d} &= \int_0^{2\pi} \dots \int_0^{2\pi} \left(\tilde{Q}_{X^1[1]}^1 + \dots + \tilde{Q}_{X^d[1]}^d \right) \frac{d^m \theta}{(2\pi)^m} = \\ &= \int_0^{2\pi} \dots \int_0^{2\pi} \sum_{l_1, \dots, l_d} \left(\tilde{\Pi}_{\boldsymbol{\rho}^i[0]}^{(l_1 \dots l_d)} A_{l_1 X^1 \dots l_d X^d[1]}^i + \tilde{\Pi}_{\boldsymbol{\rho}^i[1]}^{(l_1 \dots l_d)} A_{l_1 X^1 \dots l_d X^d[0]}^i + \right. \\ &\quad \left. + \tilde{\Pi}_{\boldsymbol{\phi}^j[0]}^{(l_1 \dots l_d)} B_{l_1 X^1 \dots l_d X^d[1]}^j + \tilde{\Pi}_{\boldsymbol{\phi}^j[1]}^{(l_1 \dots l_d)} B_{l_1 X^1 \dots l_d X^d[0]}^j \right) \frac{d^m \theta}{(2\pi)^m} = \\ &= \int_0^{2\pi} \dots \int_0^{2\pi} \sum_{l_1, \dots, l_d} \left(\tilde{\Pi}_{\boldsymbol{\rho}^i[0]}^{(l_1 \dots l_d)} k_1^{\gamma_1^1} \dots k_1^{\gamma_{l_1}^1} \dots k_d^{\gamma_1^d} \dots k_d^{\gamma_{l_d}^d} A_{[1] \theta^{\gamma_1^1} \dots \theta^{\gamma_{l_1}^1} \dots \theta^{\gamma_1^d} \dots \theta^{\gamma_{l_d}^d}}^i + \right. \end{aligned}$$

$$\begin{aligned}
& + \tilde{\Pi}_{\phi j [0]}^{(l_1 \dots l_d)} k_1^{\gamma_1^1} \dots k_1^{\gamma_{l_1}^1} \dots k_d^{\gamma_1^d} \dots k_d^{\gamma_{l_d}^d} B_{[1] \theta^{\gamma_1^1} \dots \theta^{\gamma_{l_1}^1} \dots \theta^{\gamma_1^d} \dots \theta^{\gamma_{l_d}^d}}^j + \\
& + \tilde{\Pi}_{\rho i [0]}^{(l_1 \dots l_d)} (\omega^\beta R_{\theta^\beta}^i)_{l_1 X^1 \dots l_d X^d [1]} + \tilde{\Pi}_{\rho i [1]}^{(l_1 \dots l_d)} \omega^\beta R_{\theta^\beta l_1 X^1 \dots l_d X^d [0]}^i + \\
& + \tilde{\Pi}_{\phi j [0]}^{(l_1 \dots l_d)} (\Omega^j + \omega^\beta \Psi_{\theta^\beta}^j)_{l_1 X^1 \dots l_d X^d [1]} + \tilde{\Pi}_{\phi j [1]}^{(l_1 \dots l_d)} \omega^\beta \Psi_{\theta^\beta l_1 X^1 \dots l_d X^d [0]}^j \Big) \frac{d^m \theta}{(2\pi)^m} = \\
& = \int_0^{2\pi} \dots \int_0^{2\pi} \left(\frac{\delta \tilde{J}}{\delta \rho^i(\boldsymbol{\theta})} \Big|_\Lambda A_{[1]}^i(\boldsymbol{\theta}, \mathbf{X}) + \frac{\delta \tilde{J}}{\delta \phi^j(\boldsymbol{\theta})} \Big|_\Lambda B_{[1]}^j(\boldsymbol{\theta}, \mathbf{X}) + \right. \\
& + \sum_{l_1, \dots, l_d} \left(\omega_{X^1}^\beta \tilde{\Pi}_{\rho i [0]}^{(l_1 \dots l_d)} l_1 R_{\theta^\beta (l_1-1) X^1 \dots l_d X^d [0]}^i + \omega_{X^1}^\beta \tilde{\Pi}_{\phi j [0]}^{(l_1 \dots l_d)} l_1 \Psi_{\theta^\beta (l_1-1) X^1 \dots l_d X^d [0]}^j + \right. \\
& \dots + \omega_{X^d}^\beta \tilde{\Pi}_{\rho i [0]}^{(l_1 \dots l_d)} l_d R_{\theta^\beta l_1 X^1 \dots (l_d-1) X^d [0]}^i + \omega_{X^d}^\beta \tilde{\Pi}_{\phi j [0]}^{(l_1 \dots l_d)} l_d \Psi_{\theta^\beta l_1 X^1 \dots (l_d-1) X^d [0]}^j \Big) + \\
& + \Omega_{X^1}^j \tilde{\Pi}_{\phi j [0]}^{(10 \dots 0)} + \dots + \Omega_{X^d}^j \tilde{\Pi}_{\phi j [0]}^{(0 \dots 01)} + \\
& + \sum_{l_1, \dots, l_d} \left(\omega_{\rho i [0]}^\beta \tilde{\Pi}_{\rho i [0]}^{(l_1 \dots l_d)} R_{\theta^\beta l_1 X^1 \dots l_d X^d [1]}^i + \omega_{\rho i [1]}^\beta \tilde{\Pi}_{\rho i [1]}^{(l_1 \dots l_d)} R_{\theta^\beta l_1 X^1 \dots l_d X^d [0]}^i + \right. \\
& + \omega_{\phi j [0]}^\beta \tilde{\Pi}_{\phi j [0]}^{(l_1 \dots l_d)} \Psi_{\theta^\beta l_1 X^1 \dots l_d X^d [1]}^j + \omega_{\phi j [1]}^\beta \tilde{\Pi}_{\phi j [1]}^{(l_1 \dots l_d)} \Psi_{\theta^\beta l_1 X^1 \dots l_d X^d [0]}^j \Big) \Big) \frac{d^m \theta}{(2\pi)^m}
\end{aligned}$$

We can see that the last four terms in the expression above represent the integral of the value $\omega^\beta \partial \tilde{P}_{[1]} / \partial \theta^\beta$ and so are equal to zero. It's not difficult to see now that the expression $\langle \tilde{P} \rangle_T - \langle \tilde{Q}^1 \rangle_{X^1} - \dots - \langle \tilde{Q}^d \rangle_{X^d}$ can be written in the form:

$$\begin{aligned}
& \int_0^{2\pi} \dots \int_0^{2\pi} \left(\frac{\delta \tilde{J}}{\delta \rho^i(\boldsymbol{\theta})} \Big|_\Lambda (R_T^i - A_{[1]}^i) + \frac{\delta \tilde{J}}{\delta \phi^j(\boldsymbol{\theta})} \Big|_\Lambda (\Psi_T^j - B_{[1]}^j) \right) \frac{d^m \theta}{(2\pi)^m} + \\
& + \left(k_{qT}^\beta - \omega_{X^q}^\beta \right) \frac{\partial \tilde{J}}{\partial k_q^\beta} \Big|_\Lambda + (p_{qT}^j - \Omega_{X^q}^j) \frac{\partial \tilde{J}}{\partial p_q^j} \Big|_\Lambda
\end{aligned}$$

The last two terms of the above expression are obviously equal to zero according to relations (1.13). As for the first term, we can see that it represents the inner product of the variation derivative of the functional \tilde{J} on Λ with the first ϵ -discrepancy of system (1.9) after the substitution of the main term (1.10). Since the variation derivative of the functional \tilde{J} on Λ represents a regular left eigen-vector of the corresponding linear operator $\hat{L}_{[\mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{p}_1, \dots, \mathbf{p}_d, \mathbf{U}]}$ with zero eigen-value, we can claim that it is given by a linear combination of the corresponding vectors $\boldsymbol{\kappa}_{[\mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{p}_1, \dots, \mathbf{p}_d, \mathbf{U}]}^{(q)}(\boldsymbol{\theta})$ on the complete regular family Λ . We can see then that the first term of the above expression is equal to zero view relations (1.12).

Lemma 2.3 is proved.

Using Lemma 2.3 we can replace in fact the Whitham system (1.12) - (1.14) by the equivalent system

$$\begin{aligned}
S_T^\alpha &= \omega^\alpha(\mathbf{S}_\mathbf{x}, \boldsymbol{\Sigma}_\mathbf{x}, \mathbf{U}) \quad , \quad \Sigma_T^j = \Omega^j(\mathbf{S}_\mathbf{x}, \boldsymbol{\Sigma}_\mathbf{x}, \mathbf{U}) \quad , \\
U_T^\gamma &= \langle Q^{1\gamma} \rangle_{X^1} + \dots + \langle Q^{d\gamma} \rangle_{X^d} \quad ,
\end{aligned} \tag{2.26}$$

$\gamma = 1, \dots, m + n_2 + s$, for the coordinates

$$(\mathbf{S}(\mathbf{X}), \boldsymbol{\Sigma}(\mathbf{X}), U^1(\mathbf{X}), \dots, U^{m+n_2+s}(\mathbf{X}))$$

on the submanifold \mathcal{K} .

Let us note also that in the case when the values

$$(\mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{p}_1, \dots, \mathbf{p}_d, U^1, \dots, U^{m+n_2+s})$$

give the full set of parameters on Λ (excluding the initial phase shifts), it is not difficult to get from Proposition 2.1 and Lemma 2.1 that the linear span of variation derivatives of the functionals J^γ , $\gamma = 1, \dots, m + n_2 + s$, contain all the regular left eigen-vectors $\boldsymbol{\kappa}_{[\mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{p}_1, \dots, \mathbf{p}_d, \mathbf{U}]}^{(q)}(\boldsymbol{\theta})$ of the operator $\hat{L}_{[\mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{p}_1, \dots, \mathbf{p}_d, \mathbf{U}]}$, corresponding to the zero eigen-value.

The question studied in this paper can in fact be formulated as follows: do the ϵ -terms of the Poisson brackets of the functionals

$$(\mathbf{S}(\mathbf{X}), \boldsymbol{\Sigma}(\mathbf{X}), U^1(\mathbf{X}), \dots, U^{m+n_2+s}(\mathbf{X}))$$

on \mathcal{K} give a Poisson structure for the Whitham system (2.26)? It appears that this question can be associated actually with the procedure of the Dirac restriction of the Poisson bracket on a submanifold ([28, 33, 34]). Thus, the positive answer to this question depends on the resolvability of the systems

$$\begin{aligned} \hat{B}_{[0]}^{ij}(\mathbf{X}) \beta_{j[\mathbf{q}]} \left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}, \mathbf{X} \right) &= \{g^i(\boldsymbol{\theta}, \mathbf{X}), J_{[\mathbf{q}]} \}_{|\mathcal{K}[1]} , \\ \hat{B}_{[0]}^{ij}(\mathbf{X}) \alpha_{j[\mathbf{p}]} \left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}, \mathbf{X} \right) &= \{g^i(\boldsymbol{\theta}, \mathbf{X}), \Sigma_{[\mathbf{p}]} \}_{|\mathcal{K}[1]} , \end{aligned} \quad (2.27)$$

at every \mathbf{X} , where

$$\begin{aligned} \hat{B}_{[0]}^{ij}(\mathbf{X}) &= \sum_{l_1, \dots, l_d} B_{(l_1 \dots l_d)}^{ij} \left(\frac{\mathbf{S}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}, \mathbf{X} \right) \times \\ &\quad \times k_1^{\alpha_1^1}(\mathbf{X}) \dots k_1^{\alpha_{l_1}^1}(\mathbf{X}) \dots k_d^{\alpha_d^1}(\mathbf{X}) \dots k_d^{\alpha_{l_d}^d}(\mathbf{X}) \frac{\partial}{\partial \theta^{\alpha_1^1}} \dots \frac{\partial}{\partial \theta^{\alpha_{l_1}^1}} \dots \frac{\partial}{\partial \theta^{\alpha_d^d}} \dots \frac{\partial}{\partial \theta^{\alpha_{l_d}^d}} \end{aligned}$$

is the Hamiltonian operator (1.18) on the family Λ , and the right-hand parts of systems (2.27) are given by the first non-vanishing terms of the brackets of constraints with the functionals $J_{[\mathbf{q}]}$, $\Sigma_{[\mathbf{p}]}$ on \mathcal{K} , having the order $O(\epsilon)$ according to (2.24) - (2.25).

The resolvability of systems (2.27) obviously depends on the properties of the operator $\hat{B}_{[0]}^{ij}(\mathbf{X})$ on the torus \mathbb{T}^m . Easy to see that for generic values of $(\mathbf{k}_1(\mathbf{X}), \dots, \mathbf{k}_d(\mathbf{X}))$ the foliation leaves defined by the set $\{\mathbf{k}_q(\mathbf{X})\}$ are everywhere dense in \mathbb{T}^m . However, we can see that for special values of $\mathbf{k}_q(\mathbf{X})$ the closures of orbits of the abelian group generated by the set of constant vector fields $(\mathbf{k}_1(\mathbf{X}), \dots, \mathbf{k}_d(\mathbf{X}))$ on \mathbb{T}^m can define tori of lower dimensions $\mathbb{T}^k \subset \mathbb{T}^m$. Let us denote here by \mathcal{M} the subset in the space of parameters on Λ corresponding to the generic case $\mathbb{T}^k = \mathbb{T}^m$ for the corresponding values of \mathbf{k}_q . It is easy to see that the subset \mathcal{M} has the full measure in the space of parameters on Λ .

In general case, the operator $\hat{B}_{[0]}^{ij}$ has a finite number of “regular” eigen-vectors with zero eigen-values, smoothly depending on the parameters on Λ . However, for special values of parameters

the set of linearly independent eigen-vectors of $\hat{B}_{[0]}^{ij}$ with zero eigen-values can be infinite, which is connected, in particular, with the dimension of the closures of foliation leaves defined by the set $\{\mathbf{k}_q\}$.

Easy to see that, according to our definition of the quasiperiodic function and Definition 2.2, the vectors

$$v_i^{(l)}(\boldsymbol{\theta} + \boldsymbol{\theta}_0, \mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{p}_1, \dots, \mathbf{p}_d, \mathbf{U}) = \sum_{\nu=1}^Q \gamma_\nu^l(\mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{p}_1, \dots, \mathbf{p}_d, \mathbf{U}) \left. \frac{\delta J^\nu}{\delta \varphi^i(\boldsymbol{\theta})} \right|_\Lambda \quad (2.28)$$

$l = 1, \dots, s$, represent the regular eigen-vectors of the operator $\hat{B}_{[0]}^{ij}$ on Λ , corresponding to the zero eigen-value. It is not difficult to see also that on the set \mathcal{M} vectors (2.28) are the only linearly independent kernel vectors of $\hat{B}_{[0]}^{ij}$ smoothly depending on $\boldsymbol{\theta}$. Thus, we can see that vectors (2.28) give in fact the full set of the linearly independent regular kernel vectors of the operator $\hat{B}_{[0]}^{ij}$ on Λ .

Using relations (2.28) we can claim that the right-hand parts of systems (2.27) are automatically orthogonal to the regular kernel vectors of the operator $\hat{B}_{[0]}^{ij}(\mathbf{X})$ on Λ in the presence of a complete Hamiltonian set of the first integrals (I^1, \dots, I^Q) . Indeed, according to (2.23), the convolution of any Poisson bracket $\{g^i(\boldsymbol{\theta}, \mathbf{X}), F\}|_{\mathcal{K}}$ with the variation derivatives $\delta U^\gamma(\mathbf{Z})/\delta \varphi^i(\boldsymbol{\theta}, \mathbf{X})$, $\gamma = 1, \dots, m + n_2 + s$, on \mathcal{K} are identically equal to zero. It's not difficult to get then, that for $F = J_{[\mathbf{q}]}$ or $F = \Sigma_{[\mathbf{p}]}$ this property gives in the main order in ϵ the orthogonality of the right-hand parts of (2.27) to the corresponding variation derivatives (2.4) at every given \mathbf{X} . As we mentioned already, the variation derivatives of the corresponding functionals J^γ give in fact the maximal linearly independent subset in the space generated by (2.4), so we get actually the same property for all the vectors (2.4). From relations (2.28) we get then the analogous property for the vectors $\mathbf{v}^{(l)}(\boldsymbol{\theta}, \mathbf{X})$.

In particular, we can claim that systems (2.27) are always resolvable in the case of one-phase regular Hamiltonian family Λ with arbitrary number of pseudo-phases. Indeed, the operators $\hat{B}_{[0]}^{ij}(\mathbf{X})$ represent in this case skew-symmetric operators with regular spectra, such that the nonzero eigen-values of $\hat{B}_{[0]}^{ij}(\mathbf{X})$ are separated from zero.

It can be noted also that in some examples the operators $\hat{B}_{[0]}^{ij}(\mathbf{X})$ do not contain a differential part and reduce to ultralocal operators acting separately at different points of \mathbb{T}^m . The corresponding systems (2.27) represent in this case pure algebraic systems and are usually trivially solvable. The consideration of the multi-phase situation is not different then from the single-phase one in the case of existence of the multi-phase solutions. Nevertheless, in the general case operators $\hat{B}_{[0]}^{ij}(\mathbf{X})$ have more complicated structure described above. As a result, systems (2.27) can be not solvable on the space of 2π -periodic in $\boldsymbol{\theta}$ functions for some “resonant” values of the parameters on Λ .

Let us formulate the Theorem which permits actually not to separate the single-phase and the multi-phase cases in the known examples.

Theorem 2.1.

Let system (1.1) be a local Hamiltonian system generated by the functional (1.19) according to the Hamiltonian structure (1.18). Let Λ be a regular Hamiltonian family of m -phase solutions of system (1.1) with n_2 pseudo-phases and (I^1, \dots, I^Q) represent a complete Hamiltonian set of commuting integrals (1.20) for this family, invariant under the action of the pseudo-phase group.

Let the parameter space \mathbf{U} contain a dense set $\mathcal{S} \subset \mathcal{M}$ where the systems (2.27) are solvable on the space of the smooth 2π -periodic in each θ^α functions. Then:

1) *The bracket*

$$\begin{aligned}
\{S^\alpha(\mathbf{X}), S^\beta(\mathbf{Y})\} &= 0 \quad , \quad \{\Sigma^j(\mathbf{X}), \Sigma^l(\mathbf{Y})\} = 0 \quad , \quad \{S^\alpha(\mathbf{X}), \Sigma^l(\mathbf{Y})\} = 0 \quad , \\
\{S^\alpha(\mathbf{X}), U^\gamma(\mathbf{Y})\} &= \omega^{\alpha\gamma}(\mathbf{S}_\mathbf{X}, \mathbf{\Sigma}_\mathbf{X}, \mathbf{U}(\mathbf{X})) \delta(\mathbf{X} - \mathbf{Y}) \quad , \\
\{\Sigma^j(\mathbf{X}), U^\gamma(\mathbf{Y})\} &= \Omega^{j\gamma}(\mathbf{S}_\mathbf{X}, \mathbf{\Sigma}_\mathbf{X}, \mathbf{U}(\mathbf{X})) \delta(\mathbf{X} - \mathbf{Y}) \quad , \\
\{U^\gamma(\mathbf{X}), U^\rho(\mathbf{Y})\} &= \langle A_{10\dots 0}^{\gamma\rho} \rangle(\mathbf{S}_\mathbf{X}, \mathbf{\Sigma}_\mathbf{X}, \mathbf{U}(\mathbf{X})) \delta_{X^1}(\mathbf{X} - \mathbf{Y}) + \dots + \\
&+ \langle A_{0\dots 01}^{\gamma\rho} \rangle(\mathbf{S}_\mathbf{X}, \mathbf{\Sigma}_\mathbf{X}, \mathbf{U}(\mathbf{X})) \delta_{X^d}(\mathbf{X} - \mathbf{Y}) + \\
&+ [\langle Q^{\gamma\rho p} \rangle(\mathbf{S}_\mathbf{X}, \mathbf{\Sigma}_\mathbf{X}, \mathbf{U}(\mathbf{X}))]_{X^p} \delta(\mathbf{X} - \mathbf{Y}) \quad , \quad \gamma, \rho = 1, \dots, m+s
\end{aligned} \tag{2.29}$$

obtained with the aid of the functionals (I^1, \dots, I^Q) , satisfies the Jacobi identity.

2) *The averaged Hamiltonian structure is invariant with respect to the choice of the functionals $(I^1, \dots, I^{m+n_2+s})$ among the set (I^1, \dots, I^Q) (and the choice of the set (I^1, \dots, I^Q)).*

The proof of Theorem 2.1 coincides in detail with the proofs of Theorems 3.1 and 3.2 in [34], given in the absence of the pseudo-phases. Let us say, that the considerations represented in [34] can be repeated without substantial changes also in the presence of the pseudo-phases considered in the way described above. So, let us omit here the proof of Theorem 2.1 and just make a reference to [33, 34].

Thus, Theorem 2.1 gives us a possibility to generalize the bracket averaging procedure to the case of the presence of the pseudo-phases.

Using Theorem 2.1, it is easy to prove that the Whitham system (2.26) is Hamiltonian with respect to the averaged Poisson bracket (2.29) with the Hamiltonian functional

$$H_{av} = \int \langle P_H \rangle(\mathbf{S}_\mathbf{X}, \mathbf{\Sigma}_\mathbf{X}, \mathbf{U}(\mathbf{X})) d^d X \tag{2.30}$$

Indeed, including the Hamiltonian H in the set of the functionals $(I^1, \dots, I^{m+n_2+s})$ it is easy to see that the functional (2.30) generates system (2.26) according to bracket (2.29).

Finally, we consider here just a very simple example of the generalized nonlinear Shrödinger equation in $d \geq 1$ dimensions:

$$i\psi_t = \Delta\psi + V'(|\psi|^2)\psi \tag{2.31}$$

Equation (2.31) is Hamiltonian with respect to the Poisson bracket

$$\{\psi(\mathbf{x}), \bar{\psi}(\mathbf{y})\} = i\delta(\mathbf{x} - \mathbf{y}) \quad , \quad \{\psi(\mathbf{x}), \psi(\mathbf{y})\} = 0 \quad , \quad \{\bar{\psi}(\mathbf{x}), \bar{\psi}(\mathbf{y})\} = 0 \tag{2.32}$$

with the local Hamiltonian functional

$$H = \int P_H(\mathbf{x}) d^d x = \int (\nabla\psi \nabla\bar{\psi} - V(|\psi|^2)) d^d x \tag{2.33}$$

Bracket (2.32) has d momentum functionals

$$I_q = \int P_q(\mathbf{x}) d^d x = \frac{i}{2} \int (\psi \bar{\psi}_{x^q} - \psi_{x^q} \bar{\psi}) d^d x \tag{2.34}$$

and the “particle number” functional

$$N = \int P_N(\mathbf{x}) d^d x = \int \psi \bar{\psi} d^d x \quad (2.35)$$

commuting with the Hamiltonian (2.33) and with each other. Bracket (2.32) is non-degenerate, and in general the functionals (2.33) - (2.35) represent the full set of local integrals of system (2.31) for any $d \geq 1$.

Equation (2.31) has a natural family of “two-phase” solutions given by the product of two periodic functions:

$$\psi(\mathbf{x}, t) = e^{i(\mathbf{p}\mathbf{x} + \Omega t + \tau_0)} \Phi(\mathbf{k}\mathbf{x} + \omega t + \theta_0) \quad (2.36)$$

where the complex-valued function $\Phi(\theta)$ satisfies the equation

$$(\Omega - \mathbf{p}^2) \Phi + i(2\mathbf{p}\mathbf{k} - \omega) \Phi_\theta + \mathbf{k}^2 \Phi_{\theta\theta} + V'(|\Phi|^2) \Phi = 0 \quad (2.37)$$

Equation (2.37) is equivalent to the system

$$\begin{aligned} i(2\mathbf{p}\mathbf{k} - \omega) \Phi \bar{\Phi} + \mathbf{k}^2 (\Phi_\theta \bar{\Phi} - \Phi \bar{\Phi}_\theta) &= iA, \\ (\Omega - \mathbf{p}^2) \Phi \bar{\Phi} + \mathbf{k}^2 \Phi_\theta \bar{\Phi}_\theta + V(|\Phi|^2) &= B, \end{aligned}$$

where A and B are two real constants which are fixed by the 2π -periodicity conditions for the functions $\text{Re } \Phi$ and $\text{Im } \Phi$. After fixing of the constant “complex phase” of the function $\Phi(\theta)$, the form of the functions $\Phi(\theta)$ is parametrized by the three parameters $(\mathbf{k}^2, 2\mathbf{p}\mathbf{k} - \omega, \Omega - \mathbf{p}^2)$ and is the same for all $d \geq 1$.

The form of the two-phase solutions $\psi(\mathbf{x}, t)$ depends on the $2d + 2$ parameters $(k_1, \dots, k_d, \omega, p_1, \dots, p_d, \Omega)$. It is very well known that the properties of the corresponding solutions depend on the form of the potential $V(|\psi|^2)$. We will consider here just the possibility of the averaging of bracket (2.32) on the full family Λ of these solutions.

It is easy to see that system (2.31) can be represented as a system with a pseudo-phase. Indeed, putting $\psi = \rho e^{i\phi}$ we can represent (2.31) in the form:

$$\rho_t = 2\nabla\rho \nabla\phi + \rho \Delta\phi, \quad \phi_t = -\frac{\Delta\rho}{\rho} + (\nabla\phi)^2 - V'(\rho^2)$$

The Poisson bracket (2.32) and the functionals (2.33) - (2.35) can then be written as:

$$\{\rho(\mathbf{x}), \phi(\mathbf{y})\} = -\frac{1}{2\rho(\mathbf{x})} \delta(\mathbf{x} - \mathbf{y}), \quad \{\rho(\mathbf{x}), \rho(\mathbf{y})\} = 0, \quad \{\phi(\mathbf{x}), \phi(\mathbf{y})\} = 0 \quad (2.38)$$

$$H = \int ((\nabla\rho)^2 + \rho^2 (\nabla\phi)^2 - V(\rho^2)) d^d x, \quad I_q = \int \rho^2 \phi_{x^q} d^d x, \quad N = \int \rho^2 d^d x \quad (2.39)$$

The pseudo-phase group is acting in evident way: $\rho(\mathbf{x}) \rightarrow \rho(\mathbf{x})$, $\phi(\mathbf{x}) \rightarrow \phi(\mathbf{x}) + \tau_0$, and it is easy to see that the bracket (2.38) and the functionals (2.39) are invariant under the action of the pseudo-phase group.

The corresponding solutions (2.36) can now be represented in the form (1.4) - (1.5):

$$\begin{aligned} \rho(\mathbf{x}, t) &= R(k_1 x^1 + \dots + k_d x^d + \omega t + \theta_0, \mathbf{U}), \\ \phi(\mathbf{x}, t) &= \Psi(k_1 x^1 + \dots + k_d x^d + \omega t + \theta_0, \mathbf{U}) + p_1 x^1 + \dots + p_d x^d + \Omega t + \tau_0 \end{aligned} \quad (2.40)$$

where the functions $(R(\theta), \Psi(\theta))$ satisfy the system:

$$\begin{aligned}\omega R_\theta &= 2\mathbf{k}^2 R_\theta \Psi_\theta + 2\mathbf{k} \mathbf{p} R_\theta + \mathbf{k}^2 R \Psi_{\theta\theta} , \\ \Omega + \omega \Psi_\theta &= -\mathbf{k}^2 R_{\theta\theta}/R + \mathbf{k}^2 \Psi_\theta^2 + 2\mathbf{k} \mathbf{p} \Psi_\theta + \mathbf{p}^2 - V'(R^2)\end{aligned}\tag{2.41}$$

So, solutions (2.36) can be considered here as a family of one-phase solutions with one pseudo-phase. Using relations (2.41) it is not difficult to construct the corresponding operator $\hat{L}_{[\mathbf{U}, \boldsymbol{\theta}_0]}$ and to prove that the family (2.40) represents a regular Hamiltonian family of one-phase solutions with one pseudo-phase. It is not difficult to check also that the functionals (2.39) provide a coordinate system on any submanifold given by the constraints $p_1 = \text{const}, \dots, p_d = \text{const}$ in the space of parameters, and that (2.39) represent a complete Hamiltonian set of commuting integrals according to Definition 2.2.

Thus, we can claim that the procedure of constructing of the averaged Poisson bracket on the family (2.40) is well justified in our case.

For the construction of the averaged Poisson bracket we can use just two integrals from the set (2.39). All the calculations can in fact be made in the initial coordinates $\psi(\mathbf{x})$ for the bracket (2.32). It is most convenient to take the integral N and one of the integrals I_q to construct the multi-dimensional averaged bracket. For the Poisson brackets of the densities $P_N(\mathbf{x}), P_q(\mathbf{x})$ we get the following relations:

$$\begin{aligned}\{P_N(\mathbf{x}), P_N(\mathbf{y})\} &= 0 , \quad \{P_N(\mathbf{x}), P_q(\mathbf{y})\} = P_N(\mathbf{x}) \delta_{x^q}(\mathbf{x} - \mathbf{y}) + P_{N,x^q} \delta(\mathbf{x} - \mathbf{y}) , \\ \{P_q(\mathbf{x}), P_q(\mathbf{y})\} &= 2 P_q(\mathbf{x}) \delta_{x^q}(\mathbf{x} - \mathbf{y}) + P_{q,x^q} \delta(\mathbf{x} - \mathbf{y})\end{aligned}$$

It is easy to get also the relations $\omega_N = 0, \Omega_N = 1, \omega_q = k_q = S_{X^q}, \Omega_q = p_q = \Sigma_{X^q}$ for the frequencies corresponding to the flows generated by the functionals N and I_q on the family Λ .

As a result, we define the averaged Poisson bracket on the space of fields $(S(\mathbf{X}), \Sigma(\mathbf{X}), U^1(\mathbf{X}), U^2(\mathbf{X}))$:

$$\begin{aligned}\{S(\mathbf{X}), S(\mathbf{Y})\} &= \{\Sigma(\mathbf{X}), \Sigma(\mathbf{Y})\} = \{S(\mathbf{X}), \Sigma(\mathbf{Y})\} = 0 , \\ \{S(\mathbf{X}), U^1(\mathbf{Y})\} &= 0 , \quad \{\Sigma(\mathbf{X}), U^1(\mathbf{Y})\} = \delta(\mathbf{X} - \mathbf{Y}) , \\ \{S(\mathbf{X}), U^2(\mathbf{Y})\} &= S_{X^q} \delta(\mathbf{X} - \mathbf{Y}) , \quad \{\Sigma(\mathbf{X}), U^2(\mathbf{Y})\} = \Sigma_{X^q} \delta(\mathbf{X} - \mathbf{Y}) , \\ \{U^1(\mathbf{X}), U^1(\mathbf{Y})\} &= 0 ,\end{aligned}\tag{2.42}$$

$$\{U^1(\mathbf{X}), U^2(\mathbf{Y})\} = U^1(\mathbf{X}) \delta_{X^q}(\mathbf{X} - \mathbf{Y}) + U_{X^q}^1 \delta(\mathbf{X} - \mathbf{Y}) ,$$

$$\{U^2(\mathbf{X}), U^2(\mathbf{Y})\} = 2 U^2(\mathbf{X}) \delta_{X^q}(\mathbf{X} - \mathbf{Y}) + U_{X^q}^2 \delta(\mathbf{X} - \mathbf{Y}) ,$$

where $U^1 \equiv \langle P_N \rangle, U^2 \equiv \langle P_q \rangle$.

It is not difficult to check by direct calculation that after the introduction of the action variables

$$Q_1(\mathbf{X}) = (U^2(\mathbf{X}) - \Sigma_{X^q} U^1(\mathbf{X})) / S_{X^q} , \quad Q_2(\mathbf{X}) = U^1(\mathbf{X})$$

the bracket (2.42) acquires the canonical form. It is easy to check also that the action variables represent in fact the same functionals for all the brackets (2.42) with different $q = 1, \dots, d$. Thus,

all the brackets (2.42) represent in fact the same bracket in different coordinates. It can be also checked that the averaging procedure gives also the same bracket for any other choice of the pair of functionals from the set (2.39). The Whitham system is generated by the Hamiltonian functional

$$H_{av} = \int \langle P_H \rangle d^d X \equiv \int \langle P_H \rangle (S_{\mathbf{X}}, \Sigma_{\mathbf{X}}, U^1(\mathbf{X}), U^2(\mathbf{X})) d^d X$$

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References

- [1] M.J. Ablowitz, D.J. Benney., The evolution of multi-phase modes for nonlinear dispersive waves, *Stud. Appl. Math.* **49** (1970), 225-238.
- [2] V.L.Alekseev., On non-local Hamiltonian operators of hydrodynamic type connected with Whitham's equations, *Russian Math. Surveys*, **50:6** (1995), 1253-1255.
- [3] S.Yu. Dobrokhotov and V.P.Maslov., Finite-Gap Almost Periodic Solutions in the WKB Approximation. *J. Soviet. Math.*, 1980, V. 15, 1433-1487.
- [4] S. Yu. Dobrokhotov and V.P.Maslov., Multi-phase asymptotics of nonlinear partial differential equations with a small parameter, *Sov. Sci. Rev.-Math. Phys. Rev.*, Vol. 3, 1982, Overseas Publ. Association, pp. 221-311.
- [5] S. Yu. Dobrokhotov., Resonances in asymptotic solutions of the Cauchy problem for the Schrodinger equation with rapidly oscillating finite-zone potential., *Mathematical Notes*, **44:3** (1988), 656-668.
- [6] S. Yu. Dobrokhotov., Resonance correction to the adiabatically perturbed finite-zone almost periodic solution of the Korteweg - de Vries equation., *Mathematical Notes*, **44:4** (1988), 551-555.
- [7] S.Yu. Dobrokhotov, I.M. Krichever., Multi-phase solutions of the Benjamin-Ono equation and their averaging., *Math. Notes*, **49** (6) (1991), 583-594.
- [8] S.Yu. Dobrokhotov, D.S. Minenkov., Remark on the phase shift in the Kuzmak-Whitham ansatz., *Theor. and Math. Phys.*, **166**, (3) (2011) 303-316.
- [9] B.A.Dubrovin and S.P. Novikov., Hamiltonian formalism of one-dimensional systems of hydrodynamic type and the Bogolyubov - Whitham averaging method., *Soviet Math. Dokl.*, Vol. 27, (1983) No. 3, 665-669.

- [10] B.A.Dubrovin and S.P. Novikov., On Poisson brackets of hydrodynamic type., *Soviet Math. Dokl.*, Vol. 30, (1984) 651-654.
- [11] B.A. Dubrovin and S.P. Novikov., Hydrodynamics of weakly deformed soliton lattices. Differential geometry and Hamiltonian theory., *Russian Math. Survey*, **44** : 6 (1989), 35-124.
- [12] B.A. Dubrovin and S.P. Novikov., Hydrodynamics of soliton lattices, *Sov. Sci. Rev. C, Math. Phys.*, 1993, V.9. part 4. P. 1-136.
- [13] B.A. Dubrovin., Integrable systems in topological field theory, *Nucl. Phys. B* **379** (1992), 627-689.
- [14] B.A. Dubrovin., Hamiltonian formalism of Whitham - type hierarchies and topological Landau - Ginsburg models., *Comm. Math. Phys.* **145** : 1 (1992), 195-207.
- [15] B.A. Dubrovin., Integrable systems and classification of 2D topological field theories, In: "Integrable Systems", The J.-L.Verdier Memorial Conference , Actes du Colloque International de Luminy. Eds. O.Babelon, P.Cartier, Y.Kosmann-Schwarzbach, pp. 313 - 359. Birkhauser, 1993.
- [16] E.V. Ferapontov., Differential geometry of nonlocal Hamiltonian operators of hydrodynamic type, *Functional Analysis and Its Applications*, Vol. 25, No. 3 (1991), 195-204.
- [17] E.V. Ferapontov., Dirac reduction of the Hamiltonian operator $\delta^{ij} \frac{d}{dx}$ to a submanifold of the Euclidean space with flat normal connection, *Functional Analysis and Its Applications*, Vol. 26, No. 4 (1992), 298-300.
- [18] E.V. Ferapontov., Nonlocal matrix Hamiltonian operators. Differential geometry and applications, *Theor. and Math. Phys.*, Vol. 91, No. 3 (1992), 642-649.
- [19] E.V. Ferapontov., Nonlocal Hamiltonian operators of hydrodynamic type: differential geometry and applications, *Amer. Math. Soc. Transl.*, (2), 170 (1995), 33-58.
- [20] Flaschka H., Forest M.G., McLaughlin D.W., Multiphase averaging and the inverse spectral solution of the Korteweg - de Vries equation, *Comm. Pure Appl. Math.*, - 1980.- Vol. 33, no. 6, 739-784.
- [21] R. Haberman., The Modulated Phase shift for Weakly Dissipated Nonlinear Oscillatory Waves of the Korteweg-deVries Type., *Studies in applied mathematics*, **78** (1) (1988), 73-90.
- [22] R. Haberman., Standard Form and a Method of Averaging for Strongly Nonlinear Oscillatory Dispersive Traveling Waves., *SIAM Journal on Applied Mathematics* **51** (6) (1991), 1489-1798.
- [23] W.D. Hayes., Group velocity and non-linear dispersive wave propagation, *Proc. Royal Soc. London Ser. A* **332** (1973), 199-221.
- [24] I.M. Krichever., The averaging method for two-dimensional integrable equations, *Functional Analysis and Its Applications* **22**(3) (1988), 200-213.
- [25] I.M. Krichever., The τ -function of the universal Whitham hierarchy, matrix models and topological field theories., *Communications on Pure and Applied Mathematics* **47** : 4 (1994), 437-475.

- [26] Luke J.C., A perturbation method for nonlinear dispersive wave problems, *Proc. Roy. Soc. London Ser. A*, **292**, No. 1430, 403-412 (1966).
- [27] A.Ya. Mal'tsev, M.V. Pavlov., On Whitham's averaging method, *Functional Analysis and Its Applications*, **29**(1) (1995), 6-19 (1995), ArXiv: nlin/0306053.
- [28] A.Ya. Maltsev., The conservation of the Hamiltonian structures in Whitham's method of averaging, *Izvestiya, Mathematics* **63**:6 (1999), 1171-1201.
- [29] A.Ya. Maltsev., The averaging of non-local Hamiltonian structures in Whitham's method., solv-int/9910011, *International Journal of Mathematics and Mathematical Sciences*, **30**:7 (2002) 399-434.
- [30] A.Ya. Maltsev, S.P. Novikov. On the local systems Hamiltonian in the weakly nonlocal Poisson brackets, ArXiv: nlin.SI/0006030, *Physica D* 156 (2001) 53-80.
- [31] A.Ya. Maltsev., Whitham systems and deformations., *Journ. Math. Phys.* **47**, 073505 (2006), ArXiv: nlin.SI/0509033.
- [32] A.Ya. Maltsev., The deformation of the Whitham systems in the almost linear case., *Amer. Math. Soc. Transl.*, v. 224, ser. 2 (2008), 193-212, ArXiv: 0709.4618.
- [33] A.Ya. Maltsev., Whitham's method and Dubrovin - Novikov bracket in single-phase and multi-phase cases, *SIGMA* **8** (2012), 103, 54 pages, arXiv:1203.5732 .
- [34] A.Ya. Maltsev., The multi-dimensional Hamiltonian Structures in the Whitham method, *Journ. of Math. Phys.* **54** : 5 (2013), 053507, arXiv:1211.5756 .
- [35] O.I. Mokhov and E.V. Ferapontov., Nonlocal Hamiltonian operators of hydrodynamic type associated with constant curvature metrics, *Russian Math. Surveys*, **45**:3 (1990), 218-219.
- [36] O.I. Mokhov., Poisson brackets of Dubrovin - Novikov type (DN-brackets)., *Functional Analysis and Its Applications*, **22** (4) (1988), 336-338.
- [37] O.I. Mokhov., The classification of nonsingular multidimensional Dubrovin-Novikov brackets., *Functional Analysis and Its Applications*, **42** (1) (2008), 33-44.
- [38] A. C. Newell. Solitons in mathematics and physics. Society for Industrial and Applied Mathematics (1985).
- [39] S.P. Novikov, S.V. Manakov, L.P. Pitaevskii, and V.E. Zakharov., Theory of solitons. The inverse scattering method., Plemun, New York 1984.
- [40] S.P. Novikov., The geometry of conservative systems of hydrodynamic type. The method of averaging for field-theoretical systems, *Russian Math. Surveys*. **40** : 4 (1985), 85-98.
- [41] M.V.Pavlov., Elliptic coordinates and multi-Hamiltonian structures of systems of hydrodynamic type., *Russian Acad. Sci. Dokl. Math.* Vol. 50 (1995), No. 3, 374-377.
- [42] M.V.Pavlov., Multi-Hamiltonian structures of the Whitham equations, *Russian Acad. Sci. Doklady Math.*, Vol. 50 (1995) No.2, 220-223.

- [43] S.P. Tsarev., On Poisson brackets and one-dimensional Hamiltonian systems of hydrodynamic type, *Soviet Math. Dokl.*, Vol. 31 (1985), No. 3, 488-491.
- [44] S.P. Tsarev., The geometry of Hamiltonian systems of Hydrodynamic Type. The Generalized Hodograph method., *Mathematics of the USSR-Izvestiya* **37** (2) (1991), 397.
- [45] G. Whitham, A general approach to linear and non-linear dispersive waves using a Lagrangian, *J. Fluid Mech.* **22** (1965), 273-283.
- [46] G. Whitham, Non-linear dispersive waves, *Proc. Royal Soc. London Ser. A* **283** no. 1393 (1965), 238-261.
- [47] G. Whitham, Linear and Nonlinear Waves. Wiley, New York (1974).